THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

THE SEVENTH YEARBOOK

THE TEACHING OF ALGEBRA

BUREAU OF PUBLICATIONS
Teachers College, Columbia University
NEW YORK CITY
1932

Correspondence relating to and orders for additional copies of the Seventh Yearbook and earlier Yearbooks should be addressed to

BUREAU OF PUBLICATIONS, TEACHERS COLLEGE
COLUMBIA UNIVERSITY, 525 West 120th Street, New York City

Printed in the United States of America by
J. J. LITTLE AND IVES COMPANY, NEW YORK

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The National Council of Teachers of Mathematics is a national organization of mathematics teachers whose purpose is to

1. Create and maintain interest in the teaching of mathematics.
2. Keep the values of mathematics before the educational world.
3. Help the inexperienced teacher to become a good teacher.
4. Improve teachers in service by making them better teachers.
5. Raise the general level of instruction in mathematics.

Anyone interested in mathematics is eligible to membership upon payment of the annual dues of $2.00. This entitles one to receive the official journal of the National Council (*The Mathematics Teacher*). All correspondence relating to editorial and business matters should be addressed to *The Mathematics Teacher*, 525 West 120th Street, New York City.

EDITOR'S PREFACE

This is the seventh of a series of Yearbooks which the National Council of Teachers of Mathematics began to publish in 1926. They are as follows:

1. A Survey of Progress in the Past Twenty-Five Years.
2. Curriculum Problems in Teaching Mathematics.
3. Selected Topics in the Teaching of Mathematics.
4. Significant Changes and Trends in the Teaching of Mathematics Throughout the World Since 1910.
5. The Teaching of Geometry.
6. Mathematics in Modern Life.

Bound copies of all but the first one of these Yearbooks can be secured from the Bureau of Publications, Teachers College, Columbia University, New York City, for $1.75 each.

The purpose of the Seventh Yearbook is to present, in as complete form as possible in the space allotted, some of the most helpful and interesting ideas on the teaching of algebra. Special emphasis has been placed on the function concept in line with the recommendations of the National Committee on Mathematical Requirements.

I wish to express my personal appreciation as well as that of the National Council to all contributors who have helped to make this Yearbook possible.

W. D. Reeve

CONTENTS

THE TEACHING OF ALGEBRA

RECENT AND PRESENT TENDENCIES IN THE TEACHING OF ALGEBRA IN THE HIGH SCHOOLS

BY JOSEPH JABLONOWER
Fieldston School of the Ethical Culture Schools, New York City

The forces that have given rise to the present tendencies in the teaching of algebra are represented in the persons of school administrators, educational sociologists, and philosophers in education. School administrators ask, "What pupils can study algebra?" Educational sociologists ask, "Of those who can study algebra, what pupils should be permitted to study it?" Philosophers in education ask, "What basic method shall underlie the teaching of it?" The answers that have been given by those who speak with authority are not in agreement. The teacher of mathematics must choose from among them. But he will want to make his choices on the basis of as comprehensive a view as possible of the meaning and purpose of education. This view is unavoidably colored by his training and his work. And so, whatever conclusions he reaches, persons not engaged in the same field of work will be in a position to point at him an accusing finger and say of him that he is speaking from a biased point of view. Yet no man who has had training or who has a vocation can avoid the stigma. So, regardless, we proceed to give consideration to some recent contributions of the administrator, the sociologist in education, and the philosopher in education. We shall get from them such light as they have for the teacher of algebra. We shall then consider further some of the recent tendencies that have their origins among mathematicians themselves.

The discussion falls therefore under four heads:

I. Contribution of the administrator of schools.
II. Contribution of the educational sociologist.
III. Contribution of the educational philosopher.
IV. Contribution of the teacher of mathematics.

I. The Contribution of the Administrator of Schools

Failures in algebra. The administrator in education is impressed most by the appalling number of failures in algebra during the ninth year. Briggs, in his Inglis Lecture (1930), *The Great Investment in Secondary Education,* tells us: "On three successive Regents' examinations the percentages of students who had begun the subjects in New York City high schools and who passed ranged from 30 to 79 in algebra; from 21 to 86 in plane geometry" (p. 125). The situation is no doubt worse in the rural communities.

Now, this is no kind of report to make to stockholders in an investment. Superintendents are therefore hard put to it to justify the teaching of algebra to so large a portion of the high school population. Time was when the high school population constituted but a small percentage of the adolescent population, and that percentage was necessarily the most promising part of it, academically speaking. Professor Thorndike estimates it at the most as 10 per cent. To-day the percentage is considerably higher. ". . . almost one in three of the children reaching their teens in the United States enters high school."[1]

Causes of failure. Some of the failures in algebra are no doubt due to poor teaching, the failure being therefore that of the teacher rather than that of the child, although the child bears the chief burden of it; closely related to this cause is the fact that the content of freshman algebra is too remote and too unpromising to enlist the interest of even the more intelligent children. Both these difficulties are the responsibilities of teachers of mathematics. But when full account has been taken of these two factors, there remains the significant fact that perhaps only half the number of children who enter the high school should take algebra. It may well be that a freshman mathematics course of another sort may be devised—and should be devised—but it is inviting failure from the start to attempt a course in algebra with children whose intelligence level is lower than about 110 I.Q. (intelligence quotient).

Of some interest in this connection is the fact reported by Professor Counts. He found in a study of curriculum-making in public high schools that only among teachers of mathematics were a majority willing to see that fewer pupils should be studying their subject.[2]

[1] Thorndike, E. L., *The Psychology of Algebra,* p. 3. Macmillan.

[2] In the *Twenty-Sixth Yearbook of the National Society for the Study of Education,* Part I, p. 157.

Remedies. The administrator of schools has a right to make three demands on the teachers of mathematics:

(1) That they improve the teaching of algebra.
(2) That they work toward the improvement of the content in the algebra course.
(3) That they find a substitute in mathematical content for algebra, or consent to the exemption from mathematics of those children whose intelligence is demonstrably not of the kind to carry the work successfully.

As to the last of these demands nothing more will be said in this chapter, either by way of defending the point or elaborating it. The demand is reasonable and should not be denied or combated. The decision in the case of any individual child as to whether he shall or shall not take the work is a great responsibility which cannot be avoided, hazardous as it necessarily is. As to the improvement of the teaching and of the content, more will be said in a later section of this chapter.

II. The Contribution of the Educational Sociologist

The educational sociologists trying to read purpose, high social purpose, in the whole educational process are somewhat divided in their attitude toward the subject of algebra in the high schools. They want, all of them, an adjusted individual as the outcome of the process: adjusted vocationally, socially, ethically. Some, interpreting adjustment in a narrow sense, would limit the study of algebra (or any other subject) strictly to those who are reasonably certain to need it in their life work. These vocationalists do not suggest any method by which one is to determine the extent or manner in which a given pupil will, at a later time, have vocational need for a given skill or ability. In their eagerness for tangible results and practicality they overlook some important considerations.

Of what use is algebra? First, (Professor Thorndike calls attention to it) the mere fact that not more people use a given skill or field of knowledge is not proof that they would not employ it if they had it. The history of mathematics and science gives special point to this statement. Many a chapter in mathematics which developed out of a purely theoretical interest has to-day very vital significance to the man on the street. Theories of probability led

to the development of statistical theory and they find final vindication, even to the "vocationalist," in our life insurance tables. The theory of exponents finds vocational justification when it is incorporated in the mathematical tables used by the surveyor or in our compound interest tables.

Second, it is presumption that borders on blasphemy to say for a given individual that vocationally he is bound in a certain direction, and that as a consequence he is, in the name of efficiency, to be limited by the school agency to those experiences which will equip him for his vocational pursuits. We must not define him as merely a producer or consumer of food, shelter, and clothing; we have not exhausted the definition when we say that in addition he is also a member of a community or a nation. Man has the capacity to speculate as to his destiny, the meaning of life, the nature of the universe. To the student of the history of mathematics and philosophy it is not at all strange that in reference to these things, scientists and mathematicians should again, as they did in the past, play a prominent part. Whitehead, Eddington, Jeans, Einstein are names now to conjure with, and while their pronouncements have not the same immediate influence as do those of Ford or Hearst, in the long run it is from the former group rather than the latter that men will derive fundamental notions concerning the quality of living. The four mathematicians and scientists to whom reference is here made do not comprise the entire list of those who are giving us data for and methods of deriving values. No number, however large, of mathematicians and scientists alone comprise the list. The thinkers in religion and philosophy are finally necessary. But the contributions of the scientist and the mathematician have become commonplace among all philosophers even though only some philosophers are, to begin with, scientists or mathematicians. Says Professor Morris R. Cohen in his preface to *Reason and Nature:* "To Bertrand Russell's *Principles of Mathematics* I owe the greatest of all debts,—it helped me to forge the instruments of intellectual independence."

Without pretending to develop through algebra profound philosophers, and even without necessarily being equipped to understand the more recent development in mathematical theory, the teacher of algebra may rest assured that if he is himself properly equipped for his task, he is in a position to help children in some of the first necessary steps toward *an inspired and more nearly*

mature understanding of the physical, social, and spiritual environment in which they are to have their being.

There remains a third important consideration which is often overlooked by the narrow vocationalist: the truth is being forced on us as never before that the world is in flux. Social relations and industrial organizations that twenty years ago seemed to be nearly permanent have since become modified or entirely reversed, and this not alone in countries that have gone through frankly revolutionary changes. Everywhere quiet, implacable revolutions have taken place in technology and in social thinking. The individual who has been "trained efficiently" to fit an industrial and social system of twenty years ago is a tragic misfit to-day. Recognizing this fact, engineering schools like the Stevens Institute of Technology have begun to modify their courses, framed originally for specific training, into courses of general character, where underlying principles receive major emphasis. How much more must this be true in the case of children of high school age, whose capacities are first to be explored and whose vocational interests are still to be discovered!

It is one of the most grievous errors of workers in education to attempt to determine educational practice or policy by a statistical study of current practice among adults. By such procedure educators are placed in the position of putting the stamp of approval on the *status quo*, of proposing to act as agents for the perpetuation of it. Or, as Professor Bode puts it: "Statistical investigation, for example, may show that a certain number of burglaries occur annually in a given community, but it does not show whether the community needs a larger police force or more burglars. That is altogether a question of what sort of community we happen to want."[8] Education must not evade the responsibility of envisaging a nobler order of things among men and of making the schools places where children may develop into free, planful, and imaginative men and women.

III. The Contribution of the Educational Philosopher

A fundamental truth restated. There are educators who conceive of the school as an institution whose chief purpose is to give children practice in present rich and worthy living, as a preparation and a necessary condition for future worthy living. Present

[8] Bode, B. H., *Modern Educational Theories*, p. 81. Macmillan.

worthy living means for the child a sense of successful achievement in a task that he considers significant. It includes a sense of belonging to a group and sharing with that group. No future society is conceivable in which the individual can be defined except in terms of action and interaction with other individuals through and in group relationships. The schools must furnish to children the opportunity to discover themselves. This self-discovery will be in terms of abilities in certain fields of work, interests in certain fields of knowledge, readiness for certain group relations.

We proceed now to consider briefly some of the distinctive contributions in current educational theory and practice that are the outgrowth of this point of view, and to evaluate them with reference to their application to the teaching of algebra.

Individual instruction. The first among these is the plan of individual instruction, the later elaborations of which find form in the Winnetka Plan of Carleton Washburne. This plan recognizes the importance of differences among individuals and aims to avoid the tragic waste that is consequent upon mass instruction and the lock-step organization in the general run of schools. The individual work-books and instruction sheets of the Winnetka schools, the Bronxville schools, and other schools, should be known or at least sampled by all teachers who would acquaint themselves with one of the most significant experiments in modern educational work.

In this scheme, as we have already indicated, recognition is given to and advantage is taken of the fact that paces in learning differ with different individuals. It places upon the school and its agent, the teacher, the responsibility of discovering the natural or optimum pace of the individual pupil and holding him to that pace. His capacity in one subject may be above the norm for the age or group, his capacity in another subject may be below that of the age or group, and in still a third subject, he may be average. He works at the rate of which he is capable in each and his standing is based on comparisons with his own past performances and on his performance in relation to his capacity rather than with the present achievements of his contemporaries. Washburne says:

> If a certain bit of knowledge is necessary to practically every normal person, every child should have an opportunity to master it. There should not be excellent grasp for some, good for some, fair for others, and poor for still others—there should be real mastery for every child. The wide differences that are known to exist among children make it

obvious that this mastery cannot be attained by all children—or any group of children—in the same length of time and with the same amount of practice. Hence it is necessary to provide varying amounts of time and varying amounts of instructional material for different children.

To do this under ordinary school conditions, the Winnetka Schools, following Frederic Burk's lead, have developed their individual instruction technique. This consists of restating the knowledge-and-skill curriculum in terms of very definite units of achievement; preparing complete diagnostic tests to cover all of these units; and preparing self-instructive, self-corrective practice materials. When these three things have been done, it is not at all difficult to allow each child to work as long on any unit of the curriculum as is necessary to master it.[4]

Evaluation of the individual instruction plan. This instruction, it should be noted, ignores the possibilities of using the curriculum as an experience which individuals may share, and which, through the sharing, becomes the richer for the individual. Washburne, who recognizes the need of such experiences among pupils, provides them through the extracurricular and art activities.

There is no subject, however, not even algebra, in which the meanings are exhausted in one learning experience. It is as they are discussed by the learners that the meanings become rich and significant. Thus was it in the development of the mathematical principles among the mathematicians themselves (the correspondence among the mathematicians and physicists since the sixteenth century is most interesting evidence of this fact), and thus has it been whenever people have shared and exchanged ideas. The individual instruction method leaves skeletonized and formal the material with which it is the exclusive method. Particularly is this the case with the algebra material in most of the individual instruction books and sheets.

The Dalton Plan. We consider now the Dalton Plan of instruction. Miss Parkhurst, the originator of the Plan, says of it:

> Under the Dalton Plan, the pupil is given his work in the shape of a series of related jobs. The work of any job is very carefully outlined, sometimes by the teachers, often by the pupils, depending upon the kind of school. Each job corresponds to what can easily be done within a school month of 20 days.
>
> We speak of a "job" as comprising a certain number of "units" of

[4] The *Twenty-Sixth Yearbook of the National Society for the Study of Education*, Part I, p. 220.

> work. A unit of work, in quantity, approximates or corresponds to what would usually be assigned for a daily recitation in a subject. Twenty units of work would be outlined for each subject taught. If a grade's curriculum had *five subjects,* then a job would comprise 20 $\times$ 5 units or 100 units of work. A unit of work, *from the pupil's point of view,* is not a set amount of work to be done in a certain stated amount of time, nor does a unit of history, for instance, equal a unit of music or art. Pupils take as much time as they need out of the entire amount at their disposal, to do any 20 units of work of an assignment.[5]

Conferences are provided for, as are also times for presentation of new material.

> The last half-hour of the morning (forty minutes in some schools) is designated as "conference time." During this time debates, reviews, reports, etc., are given—anything which relates to the subject for which the pupils are called in conference.
>
> These conferences, coming once a week as they do (for each subject) are not considered as a time for presenting new material. They are too infrequent for that purpose. Presentations of new material are scheduled on a "Presentation Bulletin Board," and are at the call of teachers, who schedule special calls or classes in accordance with the progress and needs of individuals or groups. These appointments are classified and posted under grade headings. In this way, a teacher may call together several individuals, or an entire class, as often as necessary, as determined by the needs of the subject.[6]

The Dalton Plan aims to improve on the individual instruction plan in that it provides an opportunity for pupils to share experiences with one another and with the teacher. The instruction period is used most generally to outline the nature of the job which the pupil is under "contract" to perform. G. W. Spriggs, writing of this period in the *Mathematical Gazette* (March, 1930), says:

> We stand on a Pisgah height and survey the promised land: unlike Moses, I have explored it, but I am conducting no tour through it. Each must explore it for himself, prospecting for its treasures, though as an old timer I am at hand to help or advise our tenderfeet. My function is to provide an opportunity, with all that this may mean, and I am no longer described as teacher or instructor, but as provisor.
>
> Following this panoramic view I retire to the unobtrusive seclusion of my desk.

[5] Article of Miss Helen Parkhurst, in the *Twenty-Fourth Yearbook of the National Society for the Study of Education,* Part II.

[6] *Ibid.*

In addition to the instruction period are provided the conference periods, which are arranged for at the initiative of the instructor or of the pupils themselves. The number of pupils may be small, or it may include the entire group on the given job.

It should be noted that individual rates are abandoned, and that individuality is allowed for in these respects only: the pupil may ask for a conference with the instructor and he must budget his own time. Miss Parkhurst and her associates make no attempt to touch the character of the content. That task they frankly leave to the specialist.

Evaluation of the Dalton Plan. I submit as criticisms of the Dalton Plan the following:

1. The pupil's performance is measured in terms of the number of exercises which he works. The emphasis—and this will be especially true where the groups are large—is on the answers. The correct answer, obtained by incorrect methods, slips by and confirms the pupil in his incorrect method. The method of obtaining the answer, teachers who have employed the Plan report, is not infrequently one of submitting a fellow pupil's answers. This difficulty is brought about in large part by a second fault, namely:

2. It is unreasonable to expect children to budget a large slice of time. Twenty periods of work is a big span for children of the eighth and ninth years of school. (Miss Parkhurst advocates the method for the fourth year of elementary school.) The result is almost inevitable that children will fall in arrears, and in their panic to make up for the arrearages will not be too careful—even when honest—about the methods by which they arrive at the answers.

The difficulties to which attention is called here are not necessarily insurmountable. Only further experimentation with the Plan by teachers who are sympathetic toward it will prove finally whether, as its author claims for it, "the Dalton Plan creates new conditions of school life in which the pupils, to enjoy them, involuntarily function as individual members of a social community." [7]

An authentic contribution of the Plan is its emphasis on the duty that is ours, as teachers, to create conditions where the child may become an active learner instead of a passive pupil.

Our next concern is with the effort of persons who are occupied with a more fundamental aspect: namely, the psychology of subject

[7] *Ibid.*

matter. These educators, of whom Professor W. H. Kilpatrick is perhaps the outstanding example, call attention to the great human waste that lies in the recitation method where lessons are assigned, sometimes taught, and finally recited. From Comenius on, reformers in education have remonstrated with teachers at the futility of attempting to graft learning on the organism called "child." Real learning must grow out of purposeful activity, the purpose being genuinely that of the child. Unless the child purposes the learning, he is but giving lip service, giving the teacher his due, and he directs his best energies elsewhere, to the so-called extra-curricular activities in the schools that provide for them, and to the gang activities if the school fails to provide for them.

The project method. To the method which aims to enlist the child's whole will in the school enterprise by giving him freedom to plan and an opportunity to realize his purpose, Professor Kilpatrick applies the term "project." The project method in its first form would make all learning incidental to the self-activity of the child. It emphasizes process and conditions of creative activity on the part of children as against learning outcomes as desired by the adult world. The teacher is the adviser: according to some extreme interpretations he is a sort of glorified janitor, according to others he is required to take the initiative and to set the stage. The learning, in the latter view, is not left altogether to chance, and the acquisition of a skill may itself constitute a genuine project.

In the first flush of enthusiasm about the method, teachers saw as real only the tangible, sensible world of hammers, nails, paint-brushes, and the like. These tools were to be the legitimate school materials, and ideas were to be the outgrowth of the handling of them. But in the symposium, "Dangers and Difficulties of the Project Method and How to Overcome Them" (*Teachers College Record,* Vol. XXII, September, 1921), Professor Kilpatrick and his associates in the movement call attention to the dangers that are inherent in too naïve a faith in the process of inner development. Recognition is given here to the facts that it is necessary for the adult world to work out selected subject matter and that there must be no shrinking from assigned problems or from set tasks. And the teacher has a function to perform. "There is no spontaneous germination in the mental life. If he (the pupil) does not get the suggestion from the teacher, he gets it from somebody or something in the home or the street or from what some more vigorous fellow

pupil is doing. However, the chances are great of its being a passing and superficial suggestion, without much depth and range—in other words, not specially conducive to the developing of freedom."[8]

Evaluation of the project method. There is a place in the project method for the teacher, and there is room for learning as learning. The chief contribution of this method is its emphasis on the spirit that should characterize the schoolroom activities. The child's will must be enlisted in the learning process if any real learning is to be done.

The direct lesson for the teacher of algebra is the important fact that algebra must be made real to the child in order that his will may be enlisted. The real should not be confused with the merely tangible. As we have already pointed out there are some minds to which the algebraic relations can never be real, just as, for some minds, $2 + 2$ can never be real. But if we limit ourselves, as we should, to minds that have the capacity, the algebraic relations have reality and concreteness, in the sense that they are capable of taking on increasing meaning as the pupils develop skill in applying these relations to varied situations. The possibilities, in terms of reality, of a given formula or process are not exhausted in our first experience with them. It is as we apply them, extend them, and involve them in new associations that they grow in reality.

This has been the direction in which our textbooks in algebra have shown most marked progress. Topics are now introduced, particularly in the work of the junior high schools, not on the basis of logical relations as they would appear to the experienced adult but on the basis of situations which seem real and worth while to the pupil and to which the algebraic or other mathematical relation or skill is applicable. The job is worth doing and this is the way the job is to be done, is the keynote of the introduction of topics or units of work.

Learning also a project. But, as Professor Kilpatrick points out, there is such a thing as a learning project. Given that certain skills are to be acquired to achieve purposes in which the pupil's will is or can be enlisted, it has been found possible and economical for the pupil to undertake to drill himself until he has achieved the necessary mastery. Instead, therefore, of picking up pieces of mathematics by the wayside, as incidents to activity, the acquisition

[8] Quoted from John Dewey by Bode: *Modern Educational Theories*, p. 163.

of given skills or concepts becomes itself the center of activity. As instruments toward the developing of skills, some excellent work-books and progress books have been developed, and should be part of the equipment of every school.

IV. The Contribution of the Teacher of Mathematics

With the administrator of schools we must take the position that not all high school pupils should be required to study algebra. We must also take the position that the methods of teaching algebra to those children who study it should be greatly improved. From those educators who are experimenting with individual instruction plans we get hints as to some of the ways of improving the methods. Through the instruction books and work-sheets they have developed an approach that is less forbidding than were the textbooks of old. All textbooks are now more attractive in form than they were in an earlier day. From the Dalton Plan and kindred plans we have found that children can do a good deal of learning on their own account. From teachers who have experimented with the project method, we have learned that if problems are shrewdly set, the learning can be effectively motivated and children can become searchers after knowledge on their own account.

Professor Thorndike has warned teachers of mathematics that while the place of mathematics in the curriculum was for a long time secure because it was the only organized body of material, that security is now being threatened by the fact that other material is now taking shape and making bids for the place. Teachers of mathematics must reorganize the content, and change points of emphasis if mathematics is not to be displaced by later and more appealing rivals.

It is not reasons of prudence, such as are indicated in this warning, that are prompting the leading teachers of mathematics to work out changes in both form and content. It is rather a sensing of the waste of human energy (energy of children and teachers) that is involved if anything less good than the available best is being done. It is the realization, also, that if the initial steps are allowed to be dull and devoid of present value, mathematics may become a closed book to young men and women who have the capacity for it.

As we have already indicated, the teachers of mathematics, and especially of algebra, take such suggestions as they can from the

sociologist, the philosopher, and the psychologist. In addition they have done much to change emphasis and to revise content. Some of these changes and revisions we shall now consider briefly.

The graph and numerical trigonometry. The first of these is the introduction of the graphs. The inclusion of this topic is now common in all textbooks and courses in elementary algebra. Yet only a generation ago most books that contained such material at all gave it only as an appendix by way of a sop to the revolutionaries among teachers who insisted that the work in graphs was not only interesting but also more simple than some of the complications in notations which were then still the heart of the algebra course.

Another important change is the introduction of numerical trigonometry. This, although more recent than the introduction of the work in graphs, is now about as generally present as graphs in most books and courses of study. For this subject, too, a case had to be made out both as to the appealing character of its content and the timeliness of it for children who are beginning the study of algebra.

Notion of functionality. More important than either graphs or numerical trigonometry because it is inclusive of them, is the concept of function. For elementary purposes we may think of two sense data as being functionally related when a change in one sense datum is invariably accompanied by a change in the other. Algebra is primarily an instrument for expressing such functional relation, and the processes in algebra may be regarded as devices for expressing this functional relation in more than one way, when more than one way is needed. It is the aim of all sciences to discover and express functional relations among their data. Sciences seek to discover how "it all depends" and to express the exact nature of the dependence. Numerical trigonometry gives interesting illustrations of relation; the graph is a useful method for representing the functional relations. Hedrick's contribution, Chapter VII in *The Reorganization of Mathematics in Secondary Education,* should be read in this connection, and also Swenson's contribution in the *Fifth Yearbook of the National Council of Teachers of Mathematics.*

If algebra is to effect any useful change in attitude of the student toward his environment, social and physical, it is in the contribution that it makes by way of enabling him to see functionality (even though he know not the term function) in that environment. The

laws of physical and social sciences are formulations of the functional relations that they have discovered. To know the world is to see it in terms of functional relations.

We come now to tendencies which are still in the forming, the fates of which are still in the balance. These are (1) the changing attitude toward problem solving, (2) the organization of subject matter on a unitary basis, (3) the uses of objective tests, (4) the introduction of chapters on statistics, (5) the close correlation of algebra with physical science, and (6) the study of rounded numbers and the nature of significant figures.

Problem solving. Thorndike has given us, in his *Psychology of Algebra*, the most exhaustive study of the psychology of problem solving. He points out the need that the problem be genuine and that it have the tang of reality. But his chief contribution is his emphasis on the true nature of the problem and his consequent catalogue of problems. A problem is a task in connection with which the individual has to select his tools and processes. A problem necessarily involves novel elements, or a novel situation to which familiar elements must be applied in a novel way. This notion of the problem is broader than is the one which makes the problem synonymous with the verbal problem. While the verbal problem has its uses, it is not the whole of the story. Pupils may have difficulty with the verbal problem and yet have acceptable mastery of algebra in its more important aspect as a tool for representing quantitative and functional relations.

The problem may be, therefore, one that calls for the selection of the appropriate operation, or the recognition of the symbol, or the selection of relevant data, or the search for relevant data, or lastly, the derivation of an equation from given or discovered data. In this last connection, too, it should be borne in mind that the equation which is thus derived is not necessarily one that the beginner in algebra is able to solve. It has served its purpose, as a problem, when the student is faced with the task of applying his knowledge of the use of symbols to the task of making explicit the relations which the verbal problem describes.

It is well to bear in mind the caution of Professor Whitehead, who, in his *Introduction to Mathematics* (p. 18), says: "One of the causes of the apparent triviality of much of elementary algebra is the preoccupation of the textbooks with the solution of equations."

It is also well, in this connection, to bear in mind that while for

purposes of circumventing examination boards we may classify the types of verbal problems to which the boards are compelled to limit themselves and drill our pupils on the basis of such classification, we are not giving the children much gain in power to meet novel situations, for in the process of drilling we have taken out the element of novelty, which is the essence of the problem.

The unit. A recent introduction in the pedagogical vocabulary is the term "unit." Professor H. C. Morrison in *The Practice of Teaching in Secondary Schools* makes much of the unitary organization of subject matter, and his colleagues in the various fields have given the term specific application. Among these collaborators is Dr. E. R. Breslich. In an article, "The Unit in Mathematics" (*Junior-Senior High School Clearing House*, p. 321, February, 1931), he lists these characteristics of the unit:

(1) It is a body of closely related facts and principles so organized as to contribute to the understanding of an important aspect of the course.
(2) It must be possible to present the unit as a whole, in a form so concise as to give the learner a clearer conception of it before he undertakes to study it.
(3) The objectives must be so definitely stated that they are clear not only to the teacher but also to the pupil. The learning products must be known.
(4) All pupils properly qualified to take the course must be able to master the minimum essentials necessary and sufficient to attain complete understanding of the unit. In addition to this minimum, the unit must contain supplementary material to allow freedom in adapting the work to the individual differences of the pupils.
(5) It must be possible to devise tests which secure objective evidence of the understanding of the unit.

We submit that Dr. Breslich gives in (1), (2), and (3) a good lesson in chapter organization of any body of material. Writers of books on the unit basis have differing notions as to the closeness of relations of facts and principles, and vary in their understanding as to how big a view the learner can take in by way of a preview. It is for this reason that books on the same subject that employ the term "unit" differ among themselves in about the same way as do the books that do not use the term. Dr. Breslich provides for the presence of enrichment material for the pupil who is able to carry more than the minimum. That provision is not unique with

the "unit" organization. And finally, the requirement that the unit lend itself to objective testing would, if carried out thoroughly, exclude much from the course that no teacher, especially one so outstanding as Dr. Breslich, would omit from the course. We shall deal with this point in the next section. But before proceeding to do so, it is well to point out, as the distinctive contribution of the unit plan, that it stresses the importance of motivating the work for the pupil by the preview through which he is able to see the field that the unit covers and the new abilities or understandings to which it leads.

Tests and scales. A good survey of tests and scales, as they apply to mathematics generally, is to be found in Professor C. B. Upton's contribution in the *Report of the National Committee on Mathematical Requirements* (1923). Since then perhaps the most distinctive contributions in algebra tests have been those prepared by the Columbia Research Bureau (Forms A and B) and the Orleans Prognosis Tests. The former are merely inventories of ground covered; the latter aim to predict probable success in algebra for the pupil.

We must, in this article, limit ourselves to mention of the scales and tests in mathematics, and refer only briefly to some of their merits and their shortcomings.

The tests can, as a rule, be administered quickly. They lend themselves easily to objective rating. They cover a larger range than is possible, as a rule, in the examination of the essay type of question. But their very merits indicate that they can serve only as supplements to other methods by which judgment as to a pupil's ability and knowledge is to be arrived at: namely, the estimate of the teacher and the type of question calling for sustained application.

The limitation of this type of test lies in the fact that the questions must be so simple in character as to be answered without too great deliberation on the part of the candidate. Even in the multiple-choice type, the amount of time allowed is and should be very small. But in order so to simplify the questions, the subject or the "unit" must be atomized. What is left, then, of that comprehensive understanding which the unit plan or any other good plan seeks as the outcome of the study? The comprehensive understanding may be present or absent, for all that the objective, new-type tests can reveal. And it is reasonably certain that many teachers, when

they expect this form of test for their pupils, will naturally limit their work in class largely to drill on this simple stimulus-response basis.

The new-type tests are good as far as they go, but they cannot go far enough. And they are as good as they can be only when they are made out by teachers of mathematics who are well posted in the more recent developments in the fields of mathematics and pedagogy. For, as Professor David Eugene Smith points out in a very pertinent article on the subject,[9] "Perhaps one source of the trouble may be found in the fact that the tests have been prepared by those whose ideal of algebra is the subject as they studied it, and who have not been in a position to know the ideals which seem to be shaping future teaching."

Statistics. It is certain that with the growing importance of statistics in scientific work of any sort, the introduction of chapters on statistics in textbooks will become more general than it is at the present time, and that more difficult aspects will be given even in the elementary courses. When noted scientists speak of laws of science as being statistical approximations, it becomes necessary for all who would understand the scientists to know the meaning of averages and of correlations, and to know, also, the limitations in the meanings of these terms. Graphic representations of distributions, as they are now introduced in the textbooks, are useful and promising beginnings in this direction. Few books, however, take up the meanings of mode, median, and arithmetic mean. And even fewer consider the conditions under which one or another of these averages is the best or most useful characterization of the distribution.

Mathematics and the physical sciences. We have made frequent references to the close connection between mathematics and science. John Perry and Sir Percy Nunn in England have been leaders in the direction of making one study of mathematics and science. Some experimentation is going on in this direction in this country. Here is a virgin field for exploration and experimentation. It is a field, however, that will remain virgin a long, long time. Teachers of mathematics are likely to be too jealous of the prerogatives of their field to be willing to risk a possible reduction in status of mathematics to that of handmaiden to science. The combined course, if it is ever worked out, will have to be demonstrably

[9] "On Improving Algebra Tests," in *Teachers College Record*, March, 1923.

superior to or at least as good as the present courses in order to win the approval of College Entrance Examination Boards.

Approximate number. The approximate number—or the rounded number—takes on real meaning only when it is first considered in connection with physical measurement. The layman usually endows mathematics with mystic powers. He is willing to accept as an accurate description of physical phenomena a result of a computation carried out to ten significant figures, when the data from which the mathematical operation arose can be correct to possibly only two significant figures. It is interesting to note that John Perry makes the discussion of this topic one of the first in his *Elementary Practical Mathematics* in order that the computations that are made later may not become fantastically "accurate." American textbook writers are introducing the topic and giving increasing attention to it. It should be an essential element in every introductory course in algebra.

There is in addition a greater, more far-reaching contribution that the teacher of algebra can give. We risk a statement that is perilously naïve for an age that boasts of its sophistication: The teacher of algebra—and for that matter, the teacher in any field—can contribute his best self. This teacher at his best is not static. He levies on all fields of thought that relate to his work. He tries to find the richest contribution that those fields can offer. He enriches his own concepts in order that he may open up richer fields for his pupils. He formulates an ideal concerning the best possibilities in his pupils. In the degree to which he dedicates himself to the task of enabling his pupils to realize those best possibilities, he realizes his own best capacities. Teaching is not a one-way process. It is as each mind influences and is influenced in the schoolroom, when it does so on a high plane, that the school can come up to its full responsibility: to assure that each generation of men and women shall add its modicum of understanding, and that it shall aspire to nobler ways of living.

ALGEBRA AND MENTAL PERSPECTIVE

By JOHN P. EVERETT

Western State Teachers College, Kalamazoo, Michigan

One who would fashion a design of material consistency must have a pattern or plan and must possess skill of performance. To successful accomplishment of any kind, these two attributes seem to be universal requisites, and for ordinary work with things they are not only necessary but sufficient. It is doubtful whether the successful teaching of algebra can ever be reduced to such simple terms, because the teacher of algebra deals not with ponderable substances, but with minds, and furthermore, the teacher of algebra is more concerned with inculcating the virtues of straight thinking than with giving the individual the advantages that accrue from information.

There are levels of instruction determined by the purposes which the processes of education are designed to serve. There are certain techniques such as spelling and many of the forms of vocational proficiency in which it is possible and desirable to set up the objectives and the methods of securing the objectives in terms of definite scientific precision. There appears at present no prospect of attaining a comparable status of fixity in algebra, because, in the first place, the symbolism and forms of algebra are seldom ends in themselves, and, in the second place, the ascendant powers to be derived from a study of algebra reside in an understanding and an appreciation of relationships between ideas even more than in the ideas themselves. For instance, $3x$, 4, and 19 are suggestive of ideas in their own right. Place these same expressions in the form $3x + 4 = 19$ and there are now included some ideas of *relationship* totally foreign to and different from the expressions themselves. There is no comparable situation in arithmetic nor are there similar ideas in any other subject except as they incorporate the methods and language of this particular mathematical science.

Simplifying and clarifying thought processes. It is the mission of algebra to simplify and to clarify reasoning processes.

This is accomplished largely through the substitution of general ideas of the nature of concepts for individual statements or ideas. Thus, algebra makes an observation concerning the rectangle that is embodied in the formula $A = lw$ and by that simple expression of relationships the mind is given power to deal with all rectangles and is, at the same time, forever freed from the necessity of having to apply a special method of investigation to any particular rectangle.

The formula $A = lw$, however, carries within itself no inherent implication of its own meaning. Its significance is not like a sunbeam, or a splash of water, or a wild turnip in its intrusion upon the senses. Complete immunity from the significance of the last three after contact must be purchased, if it can be attained at all, by great effort, but immunity from the significance of $A = lw$ is not purchased by any effort whatever. It is the acquaintance with the meaning of, not the immunity from, $A = lw$ that must be purchased with a great price. The reason for the contrast is that familiarity with the physical relationships of our bodies and their environment has been gained by a million years of fierce contact, but the relationships of our mental attributes to one another and to our environment have been the subject of attention only for a period that is infinitesimal in proportion to the entire span of human experiences.

When the statement is made that one of the principal aims of algebra is to simplify and clarify thought processes, that assertion contains no implication whatever respecting the simplicity of the subject itself. As a matter of fact, there is nothing that has yet been discovered from either casual or scientific study in the subject of algebra to indicate that understanding of it can ever be easily acquired. For anyone who wishes to travel widely, transportation has been simplified by the automobile, but between the desire of the individual to travel at the rate of fifty miles an hour and the consummation of that desire there intervene such complexities of design and difficulties of accomplishment as to tax the mind and creative capacities of man in ways and to an extent that not only transcended the ability but even the imagination of our immediate ancestors. The ease with which we travel by automobile is one of the wonders of this wonderful age, but the cumulative effort that has been put forth to make this ease possible transcends by far any exertion necessary to effect previous modes of travel. In the

realm of thought algebra occupies a somewhat comparable place. It represents an advanced type of thinking that has become productive of satisfaction because it represents and reflects the capacities and the attainments of the mind in the field of thinking.

Control over the material and the spiritual forces of the universe has come about through the recognition of relationships not apparent from the study of things themselves. Man started on his inquiry into the forces of nature with an inherent, instinctive perception of motion, needing for its factual interpretation neither processes of mental analysis nor the assistance of symbolic representation. To observation and conscious perception of the external phenomenon of motion there have been added by reflection and analysis the ideas of *velocity* and *acceleration*, of *constant* and *variable*, and many laws which introduce elements of predictability and of reliability where formerly there ruled the apparent vagaries of caprice. Algebra provides the symbolism and the idealistic methodology for all such situations. Its language and its formulations may be difficult to understand and may require of him who would become its master the price of strenuous exertion, but there has not yet appeared any substitute. If this is true where merely things and their relations are involved, how much more is algebra interwoven with the understanding of the intricacies of social behavior?

Who shall study algebra? An attempt to employ the automobile and its nature to illustrate a phase of mental activity might easily leave obscure a fundamental difference between the place which the organization of physical forces occupies as contrasted with that which the organization of intellectual forces occupies. While the ordinary person enjoys all the advantages that accrue from use of the machine, it is left to the engineer, the expert designer, and the manufacturer to attend to all the difficult details of creation of the machine. As a matter of fact, an automobile behaves in about the same manner for the driver who understands every item of its composition and for the person who is almost utterly ignorant of mechanical laws. Why not leave the comprehension of algebra to the few who elect to understand it? The question is a perfectly fair one and it is not difficult to answer. It would be easily possible to confine the study of algebra to a selected few, in exactly the same manner that it is possible to place the responsibilities of government or the control of any social institution in the hands of a few, but such an arrangement is quite

abhorrent to modern ideals. To attempt to build an aristocracy of brains at the expense of the mass is quite a different proposition from that of the acceptance of specialized industry. Thought, thinking processes, and the ability to appreciate mental and spiritual accomplishments are looked upon to-day as the rightful possessions of every individual.

Problems of individual living simplified by specialization. The economic and other advantages of specialization have enormously simplified the ordinary problems of daily life. For very few of America's inhabitants does there exist either in point of difficulty or of effort the individual burden of responsibility that characterized the life of our immediate ancestors. "Life is increasingly complex" is a favorite expression of educators. How can such a statement be made in face of the fact that scarcely can there be found to-day a person who possesses the range of versatility in supplying human needs that constituted not the exceptional but the ordinary accomplishments of the pioneer? Looked at from the standpoint of the individual, life grows more simple, not more complex. Whether we contrast turning an electric light switch with lighting a tallow candle, or ordering a loaf of bread from the grocery with baking the bread at home, or adjusting the thermostat with maintenance of the fireplace, the element of greatest simplicity is overwhelmingly with the life of the person of to-day as compared with the life of the person of yesterday.

But the comparative freedom and security of the individual from the menace of forces which formerly drove him through long hours of fiercer toil, with less adequate assurance of ultimate success, have been purchased at a great price. The responsibilities formerly assumed by the individual are no less pressing now than they were of old, but to-day they have largely been taken over by society as a whole. To survive to-day no single individual needs to possess the education nor to exhibit the industry without which the colonists who originally peopled America would have perished in a single generation. The relation of to-day's individual to his own survival is different from that of his grandfather, and is less immediately insistent, but this circumstance is due to the fact that social consciousness and community solidarity have replaced the old era of individualism. Less than a hundred years ago he who essayed a clearing in the forest did so with small regard for his neighbors. Even the older countries of the world have only lately come

under the sway of forces which were not mainly encompassed by the purview of the immediate neighborhood. Where modern civilization has become established we have no local famines. The state of Arkansas may be prostrated by flood and drought, but its inhabitants do not starve. On the other hand the blight of depression also is not localized—it reaches out to encircle entire nations; it crosses international boundaries and invades the territories of neighbors who never knew that they were neighbors.

Problems of social organization increasingly complicated. Society, whether political, economic, or intellectual, is sustained today because it makes contact with the individual in a thousand ways that were unheard of a generation ago. Every mechanical invention, every social advance, every incident of political emancipation simplifies the problems of living for the individual, but by that very act complicates with increasing intricacies the problems of social organization. So the complexity of our lives is an elective affair. This age of specialization and of comfortable ease and of opportunity imposes comparatively few requirements upon the individual so far as his own needs are concerned. Under the beneficent reign of present-day social customs there are many rewards for active participation in affairs, but there are few rebounding capital penalties for neglect or refusal to assume individual responsibilities.

As furnishing the criteria for determining effective methods of teaching, the status of the individual can never change. Only the degree of our respect for the individual is subject to modification, because it has been decreed by nature that each person's education is strictly a personal affair. The statement, "All men are created equal," is as true in some respects as it is untrue in others. As to acquired characteristics, each person starts from zero. He does his own learning—nothing is inherited. The educational psychologist is mainly confronted with problems in which educational differences enter largely as determining factors. The functions of the curriculum maker must take into account social requirements, and these place the individual in a very different relative position from that which he occupies as a learner. In the latter rôle the individual stands supreme, but with respect to what he shall learn the individual assumes a position where his personal interests are seldom, if ever, of paramount importance, and where even his personal aptitudes must not often be allowed to constitute or to dominate greatly the content of his course of study.

It is easy to propose as an ideal a curriculum which shall be adjusted to the individual. There are plausible arguments in favor of the organization of as many courses of study as there are pupils. It is true that the engineer has a use for mathematics which is different from that of the musician, and the requirements of both of them differ in turn from the needs of the sociologist. Besides, the study of individual differences has progressed far enough so that it is reasonably certain that from every or any point of view there are not often encountered two persons who are alike. Individuality is one of the greatest things in the world, and no scheme of education can be effective which neglects it; but it should be noted that no genuine scheme of education can hope to prosper on a false notion of what individuality means. An oak tree, a basswood tree, and an onion possess striking attributes of those characteristics of individuality which are dependent upon differences, but their individuality is wholly static. Nurture that is appropriate to the nature of these plants promotes the particular type of individualism that belongs to each one. Education may, undoubtedly, be mainly devoted to such an aim, but a program with this aim alone is inadequate and undesirable because it overlooks and neglects the important fact that people are social beings and accordingly are concerned with community interests and advantages to such an extent that it is doubtful if any program of education whatever would be necessary or justifiable were social interests to be eliminated, or even subordinated to those that are strictly individual.

The general import of educational literature of the day seems to be in the direction of decrying uniformity and exalting individuality, apparently neglecting to observe that no one ever could have an opportunity to develop or exercise his individuality except in an atmosphere where likeness also prevailed. What we probably need is the wisdom to perceive the particulars wherein individuality should be exalted and wherein it should be suppressed. We should courageously recognize the fact that it is the function of education both to indoctrinate and to emancipate the mind. For instance, it is highly important that neighboring peoples use the same language, but this is accomplished only to the extent to which uniformity, not differences, in the meaning of language forms is attained. It is in the skillful use to which language is put that a premium is placed upon the exercise of individuality, not in the lan-

guage forms themselves. It is even necessary for people to *think* alike in order to attain national solidarity and permanence. It may safely be asserted that the margin of tolerance for the exercise of individual initiative is almost wholly determined by and dependent upon a background of knowledge and beliefs and points of view which are common to all. Freedom of thought is encouraged by the breadth and extent of the body of universally accepted ideas and attitudes and beliefs, and freedom of thought is discouraged and expression suppressed when people are deprived of the opportunity to subordinate their individual idiosyncrasies to universal ideas. Galileo and Millikan pursued their inquiries into new revelations of light under circumstances whose chief contrast consisted in the extent to which the era of each of the two men was dominated by the extent of uniformity in acceptance of the facts of nature and the similarities of mental outlook. Both of these men possessed imaginative ability to a high degree. As a creator and discoverer of new ideas Millikan's thinking was circumscribed by knowledge using laws of which Galileo and his age were ignorant. Would anyone declare that Millikan was less free than Galileo to exercise his genius because of what he knew as a result of the extent and precision of modern research? Yet every demonstrated law of nature by just that much leads directly away from individuality and toward uniformity.

The contemporary society to which Millikan ministered was tolerant because the education and thinking of his time possessed a background of common facts and methods as contrasted with individual fancies and vagaries. To become cognizant of a law circumscribes one's freedom always, but the opportunity to use the law to advantage opens to its possessor new avenues of freedom.

The mathematical needs of the engineer and of the musician are different, certainly, but they are also alike in all essentials except those which are technically associated with each of the two professions, and in an elementary course the mathematical demands of the two professions are identical. The great fundamental characteristic of mathematics—drawing of definite conclusions from specific conditions—is not different for the musician from what it is for the engineer, either with respect to the need of the ability or the nature of the ability. The result of multiplying seven by nine is not different because one person happens to be a musician and another an engineer. But, you say, the one builds bridges and the

other composes operas; the one perfects his technique at the designing table, the other upon an instrument of musical consonance; but each one is an individual before he is a technician; each one is a citizen as well as a specialist; each one's happiness and usefulness is inextricably interwoven with his social relationships. Each one occupies a niche in the world's affairs that is distinctive, but each one's nature and interests and activities represent identities many times oftener than differences.

Perspective in teaching algebra. Teachers of mathematics, like all other teachers, need to improve their methods; but they probably need, far more, to acquire a perspective which will enable them to gain a better general vision of the place that mathematics is prepared to take in the lives of people. The teacher of algebra, especially, must not allow the necessity for becoming acquainted with symbolism and forms of procedure to obscure the fact that the essential thing is to illustrate the existence of law and to encourage the ability to reach conclusions. He will probably be enthusiastic in his work to the extent to which he perceives that mathematics furnishes the only medium which the human mind has yet discovered for drawing the conclusions that necessarily follow from set conditions, and that algebra is the science which typifies and parallels the relationships of changing values which every person must recognize and deal with every day if he is to be a contributing and not a parasitical member of society. The teacher of algebra will realize that schools which teach their pupils to think are needed more than schools which emphasize chiefly the details of vocational procedure. He will recognize the fact that stimulation of the ability to appreciate things and relationships not immediately obvious to the senses is essential to young people in an age when living has been simplified by an organization of objects and forces which is complicated beyond the power of any mind to comprehend of itself, and which depends for its interpretation upon abilities arising from social inheritance rather than upon any ability of the individual or even the collective mind. The teacher of algebra will recognize that reality and sensuality are not synonymous. He will be interested in discovering and appropriating as illustrative material for his course the incidents and applications of algebra which seem to have been translated into actual practice; but he will realize that, after all, pupils need to be given free range for the exercise of their imagination, and that any situation is "practical" which

challenges the thought of a pupil, regardless of whether it can be transmuted into coin of the realm or not, and likewise regardless of whether the situation actually exists or not. The ability to cope with circumstances that are new constitutes the true measure of intellectual strength and this ability is not cultivated in an atmosphere of activities confined merely to reactions to sensory stimuli. Finally, the teacher of algebra needs to realize that the cultivation of individuality requires a sensible interpretation of the meaning of individuality. There are in algebra, and in life generally, many laws, symbols, and operations in which the implicit belief of the individual is necessary, and to which the individual should conform. Algebra is primarily a mode of thinking, offering to every inquiring mind an instrument of precision in interpreting and understanding world phenomena. It offers here and there a glimpse of eternal truth, whose beauty resides in its quality of being able to retain its permanence in a world where change is such a universal factor that different minds never react in quite the same way, and even the same mind never reacts twice with just the same implications, and where events never are repeated with exactly the same significance.

One of the most outstanding results of the cumulative accomplishments due to our social heritage resides in the extent to which symbolic language has been perfected. In almost every department of human endeavor it is possible to anticipate actual accomplishment down to the most minute detail in terms of symbols whose only relation to the actual things themselves is in the meanings which these symbols have for him who creates them and for him who reads them. Often this scheme of symbolic representation is able to reach out in a creative way far beyond the actual limits of accomplishment that have hitherto been effected, leaving to human hands the privilege of catching up with the ideals of accomplishment set by human minds. Illustrations of the power of symbolic representation are easily found in architecture and engineering, although any expressed theory of politics or economics or philosophy belongs to the same classification. Contemplation and discussion of such ideas as those associated with *justice, fear*, and *ambition* have no expression whatever in terms of their own existence. Their meaning must forever be construed wholly in terms of acts to which they apply or else in terms of artificial language to which common interpretations give significance. Although these ideas have neither form nor substance nor even existence of their

own, they, nevertheless, are genuine, real, and universal forces which dominate life no less than do the things to which our five senses give heed.

Algebra is a mental creation which is capable of asserting itself in a material situation without being, or becoming, any part of the thing itself. Thus a circular cylinder may be made out of wood, tin, or cement, or it may exist only in one's mind, but with impartiality and certainty algebra steps in to assert that, regardless of actual substance and regardless of specific dimensions, $V = \pi r^2 h$ expresses the relations of volume, radius, and altitude. A pupil is quite likely to encounter this, or any other formula, in a cursory manner, giving to V perhaps the value that follows from particular measures of r and h. (Such a formula offers a splendid opportunity for the organization of a unit of study if one desires to follow that plan.) A cylinder may vary in shape from a stretched piece of thin wire to the wafer thinness of a phonograph record. Countless, infinite, are the values that may be assumed by V, r, and h. Time or place or person matters not—always, forever, here is a law, a fact, a reality, more enduring than any monument of granite, more universal than any known property of matter in the physical universe, more truly indicative of the intellectual capacity of the mind than any discovery which sensory experience has ever inspired. Can he who once grasps such an idea ever again contemplate the world without increased respect for law? Can he who fails to understand that order reigns in the midst of unlimited variety ever hope to think in terms of the clarity and the rightness that characterize the thinking of him who knows?

He who has been exposed only to the routine of repetition or whose experience has been limited to repetition of numbers or to the performance of manipulative tasks may see in algebra "a bag of tricks." Many, alas! have gone no further. The error is a hazard of the subject, prevalent always, but avoidable generally if teachers themselves are thoroughly capable.

Development of deductive reasoning. The ability to classify things and events, to reach conclusions respecting specific items by applying the laws of generalization, which we call deductive reasoning, constitutes one of the chief instruments by which man has confirmed his superiority; and a contradictory and often baffling peculiarity of mental growth decrees that the method by which we think is not the method by which we originally learn. It is ordained that

our discoveries shall be made slowly, toilfully, little by little, and our convictions established through many experiences with related encounters. Often we accept, and properly, the facts and generalizations of others; we even accept the validity of laws which come to us with the force of convincing authority, but *thinking* and *wisdom* are different from knowledge and acquiescence in that the transcendent qualities of the former are ever the results of one's own activities. Mere exposure to facts is often sufficient to insure learning them; but the case is not so with thinking—that is an activity which thrives and develops only through its own exercise by every separate mind.

Algebra has one great central purpose—to enable one to reach conclusions which are consistent with hypotheses, and to arrive at this attainment there are levels of effort which are different and distinctive in quality and nature. Much of the fundamental language of algebra is arbitrary in fact and nearly all of it is arbitrary in effect, so far as the learner is concerned. Wherever this phase of the subject is encountered it must be learned, and consequently taught, as an outright instance of the application of pure memory. For instance, one of the algebraic inferences of the expression xy is that of multiplication between x and y. No amount of reasoning on the part of the pupil could ever lead to that conclusion. As a matter of fact the form of the expression could just as well denote addition, or subtraction, or division, or no operation at all. No responsibility rests upon the pupil for the actual meaning of the expression; all he has to do is to *accept the fact* that xy denotes a multiplication between the numbers and to *remember the fact.* Failure in the first of these essentials denotes an unsocial attitude; failure in the second denotes lack of application; but neither success nor failure in mastering the meaning of the expression implies anything whatever that is not included under the term *general intelligence.* Were the objectives of algebra confined in large part to interpretation of the kind just illustrated, the subject would have nothing distinctive to offer a pupil, and to the extent to which algebra is confined to exercises in rote learning, or the following of model forms of procedure, or the working of examples with sole regard for the answers, to that extent it is algebra in name only. Such activities contain none of the essence of mathematics, however much the language may employ the phraseology of numerical symbolism. Mathematics is not primarily the science of numbers or

of operations with numbers. Numbers and their uses are generally associated with mathematics because they constitute a satisfactory medium for expression of the kind of ideas with which mathematics deals. From a teaching standpoint it might be a good thing to declare a number moratorium for a week once a year, during which every numeral of any kind whatsoever would be rigidly excluded from all classes in mathematics. This would leave the classes in mathematics no alternative than to do some mathematical thinking. This necessity would probably be fatal alike to the so-called teacher of mathematics who is only a drillmaster in manipulative pyrotechnics and to the general educator who does not know that mathematics is anything other than the mastery of certain specific responses to given stimuli.

Practical applications of algebraic forms. The problems associated with the teaching of algebra have not been simplified by the evolution of customs and procedures which give to some algebraic forms a value and usefulness that attract the attention and win the commendation of the general educator, with his flair for the practical. Algebra was introduced into the curriculum before it had the commonly accepted applications of to-day, such as are found in the formula and the graph. It is most fortunate that during the nineteenth century the ideals of education were not so narrowly encompassed within the limits of practical demands as seriously to interfere with a curriculum content which included a large amount of material enabling the learner to enlarge the frontier of his knowledge and of his mental forms without being too much dismayed by doubt of the immediately employable attributes of those things which happened to engage his intellectual energies. When, prior to the middle of the last century, algebra was made a part of the secondary school curriculum of this country, no amount of attention to scientific methods of curriculum construction, based upon job analysis, or questionnaires, or any of the other techniques of to-day, would have included the formula. The "rule of three" and unitary analysis took care of all the situations of relationship that the man on the street or the "natural philosopher" was likely to encounter. In a laudable endeavor to eliminate useless and archaic material from the course of study and to include topics and activities of social significance, educators have profoundly modified both the outlook and the methods of the teaching profession, but it is possible, quite probable in fact, that perspective

has been sacrificed to the apparent demands of immediate utilitarianism.

A school program which is directed with more than incidental attention to utilitarian ends is subject to at least two grave defects, in spite of the fact that an appeal for the practical can be easily phrased in attractive language that renders such a program popular. The first defect of a vocational program, using the term "vocational" in either a general or a specific sense, is that such a program is based upon thinking that was appropriate to living conditions of two centuries ago. There was a time when skill and experience constituted the main reliance of old age, or at least the later years of middle age. For the great mass of people to-day no long period of apprenticeship is essential to the acquisition of proficiency whether in the shop or in the professions. A corollary of this circumstance is seen in the unfriendly attitude of industry toward aged workers. The operation of the same idea is to be found in the prevalence of age limits in the salaried professions. We deceive ourselves if we conclude that the present-day agitation for old-age pensions is primarily due to a sudden amplification of the spirit of altruism.

We are moving toward a solution of a new problem, or, if you prefer, toward the solution of an old problem clothed in the language of new conditions. Precisely the same factors that have removed from the aged worker the safeguards which formerly surrounded his old age have operated to make it easy for youth to assume a position of commanding adeptness easily and quickly. It is a waste of time and money to keep a young person in school beyond a maximum of a few weeks in order to train him for a vocation, because a few weeks of training bring to him all the expertness he will be called upon to exhibit.

The second grave defect of a program directed by practical applications is that in its endeavor to serve the requirements of to-day's apparent needs it is quite likely to neglect both the less obvious requirements of the times and the probably changed requirements of to-morrow. As has been already pointed out in this chapter, we have not solved the problem of living by education of the two-blades-of-grass-where-one-grew-before variety. The newer problems which confront us at present have much more to do with coöperative, social reactions of the individual than with the extent or quality of personal performance. Skill and precision in the consummation of an act on the part of the individual may be due wholly or in large part

to the formation of satisfactory habits. Now habit and habit formation are strictly individual matters. Several individuals may acquire the same habit, to be sure, and they may avail themselves of outside assistance, but during the formation of the habit, and throughout all its application and exercise, it remains strictly a personal possession. The benefits that accrue to society from establishing good habits in the individual reside in the quality and desirability of what is actually produced. That is the reason why sturdy habits of industry and thrift so admirably sufficed to enable our pioneer ancestors to subdue the wilderness. Those habits are still essential to good citizenship, but they will not carry a young person as far to-day as they carried his grandparents, for the simple reason that coöperative activities call for a different kind of mental alertness from that demanded by solitary performances.

Influence of social forces. In the two outstanding features of the educational literature of the past two or three decades—the prevailing emphasis upon the elimination of the obsolete and the useless from the curriculum and the inclusion of the useful—a person has been looked upon as a contributor to his own and society's welfare in terms of the development of his own innate capacities, while the necessity for recognizing social forces which operate outside the individual has scarcely been taken into account. Psychology of the stimulus-response school accounts for a good deal of human and animal behavior, but it fails utterly to explain the human activities which are directed by influences which operate not *on* minds but *between* minds. In our psychological thinking and experimentation we have gone far enough to take care of the situations comparable to those wherein a herd of buffaloes is attracted toward water or flees from danger. The fact that they act alike is no evidence either of coöperation or of thinking. Nor is a division of labor among men, based upon physical prowess or the accident of location, any evidence of thinking or of coöperation. Specialization may proceed far and still be explained on the basis of a stimulus-response psychology. There comes a stage in the development of social institutions where thinking and reasoning include something beyond and outside the formation, or the coördination, or the organization of habits. That point is reached when two or more persons come together and deliberately weigh the advantages of specialization.

It happens that just now the world is more concerned with the

disadvantages than with the advantages of specialization. Both advantages and disadvantages present problems of the same kind, calling for an evaluation of consequences that follow not so much from acts as from points of view. Most of the questions which arise in connection with universal education, freedom of church and state, the capitalistic organization of industry, the administration of justice, and the like, deal with points of view wherein the one dominating principle that prevents utter chaos is that of consistency in man's contemplation of causes and consequences. Nature imposes no limitations upon the imagination of man; and then decrees that he shall profit or suffer from changes which he effects in his environment according to whether he regulates wisely the modifications which he produces. Every step which has raised man above the level of animal existence has come about as the result of his ability to establish—consciously or unconsciously—the correct relations between the sowing and the harvest. It is not enough merely to *maintain* this relationship. The oak tree does that; it even evolves to a higher from a lower form of organism. It does so solely as the result of its environment, granting that it constitutes in itself as much a part of its environment as does anything outside itself. Man is subject to all the forces that govern the oak tree, and more. He alone of all the creatures of the earth has the ability to make and comprehend a *plan*. Not only can he follow an urge, he can create an urge and that, too, as the result of vicarious impulses.

The adequacy of any plan, whether of the level of making a living or of social adjustments, is never determined by the plan itself; there enters always the human factor. This has been true even in communities governed by the most absolute monarchies. Many persons think that the success of democratic government depends upon the ability and the willingness of the people to respect wise leadership. Such an outcome, however, will be a strictly coöperative affair in which all the people will share, not merely in the relationship of leaders and followers, but also in the capacity of intelligent contributors to a plan of living. There can be no permanently satisfactory exercise of the privilege of choice except by those who realize the implications of alternatives. Consequences must not only be endured, they must be anticipated. The pattern of activity must be made not merely in terms of acts, but with respect to changed conditions brought about by those acts.

We seem to be able to widen our circle of possibilities only by

reducing to law an ever increasing number of procedures, because without law, there can be no prediction of outcome. As an unswerving and universal instrument of prediction, mathematics stands without a rival. This quality resides in the power of mathematics to affect thinking no less than in its application to everyday affairs, though it is in the latter capacity that the uses of mathematics are more easily recognized.

Formalized types of behavior constitute the means by which social and intellectual advances are retained, but they do not contribute to advancement. They perform the function of a one-way brake which retards retrogressive movement but has small ability to do anything but hold gains already made. Thus the multiplication table, from being once a mere stimulator for the alert minds of those who would escape the slower, but more objective, methods of line reckoning, has itself reached a stage of more or less set perfection in which its use and possession are recognized as desirable accomplishments for every person. In the process of reaching this stage multiplication has attained a degree of mechanization which facilitates thinking rather through the medium of what it enables the mind to accomplish and the effect upon outlook than in the mental processes of the operation itself. As civilization moves along it is a law of progress that the new, alluring, perhaps even exciting and adventurous items of one day shall be the commonplaces of the next. We are perfectly familiar with this tendency in seeing the manner in which the necessities of one age grow out of the products of the imagination of a preceding age. We do not so readily perceive that thinking undergoes a precisely similar metamorphosis.

It may be argued with some degree of plausibility that in the case of a mental acquisition which represents a progressive development of mental powers, the child in his education should receive the benefit of the culture epoch theory at least to the extent of being given an opportunity to translate an accomplishment into terms of his own growth and his own thinking, but this is impossible. People are subject to laws of adaptability which are as effective in their way as are the laws of protoplasmic cell structure. What a person becomes and the way in which he thinks depend no less upon the circumstances with which he is surrounded than upon the circumstances of his biological inheritance. One may question the soundness of any or many, but not of all the customs or ideas which he encounters. Life is too short for any individual

to prove for himself by discovery or otherwise the fundamental integrity of the forms with which he is surrounded. Similarly, life is too short for any individual to evolve from within himself the evidences of progress which the race has attained by the slow process of evolution. The result of all this is that time changes the usages to which any formulation may be put, but never changes the necessity for providing a margin of thought-provoking material whose virtue resides in its generality rather than in its specific applications. As times goes on the body of apparent factual material increases, making it increasingly easier to concentrate attention upon worship of ancestral contributions and progressively more difficult to incorporate into one's thinking the elements of flexibility without which adaptability ceases to become anything more than adherence to the fixtures of the *status quo.*

It was perfectly natural that mathematics should have first had its power recognized in connection with the material adjuncts of the details of everyday life. It is, perhaps, quite as natural that it is difficult to perceive that the more remote and subtle relationships of social institutions cannot be comprehended except through the exercise of the same species of definite, consistent, and orderly analysis that has brought systematic and satisfying certainty of outcome to the tangible aspects of commodity interchange.

Mathematics came over the far rim of history with many characteristics already well established. What gave rise to the elementary forms of mathematics, like numeration, we do not know at present, and it is not likely that we shall ever know. Within the reach of historical records the evidence is abundant that in the development of mathematics, satisfactions in the realm of thinking have generally preceded satisfactions of a more prosaic nature as derived from acts and things. "What's the use?" is by no means a new question. It is a highly proper question to raise—one upon which safety largely depends, though it may be used to impede progress if the answer is based upon partial consideration of the thing involved. This is one of the chief dangers that confront our educational system to-day. Curriculum making bids fair to aspire to the status of a science and to speak with authority before it has developed a comprehensive outlook. It is comparatively easy to gain the benefits of popular acclaim by assuring people that to obtain salvation it is necessary to attain the degree of acquisition attained by everybody else, and to add the further assurance that

any effort to go farther is useless. There is something peculiarly satisfying to most of us in being told that perfection resides in what we have already attained, and if this attainment has been reached by what appears to be incidental effort, all the more compliment to us.

That education may be arrested upon a plateau of the obviously utilitarian is possible. This outcome would not be bad were the obvious and the ultimate to coincide, but the law of change which permeates every atom of the universe, forever precludes even an approximate conjunction of that which *apparently* is with that which *really* is. It is a baffling provision of nature that into the realm into which we can go—the future—no one can look for the slightest fragment of an instant, while into the realm into which we can look—the past—no one can go by the slightest fraction of a step. The confidence with which we face the future may be born of ignorance, but the serenity and security with which we experience the future is a matter of the degree to which the past has been properly evaluated in terms of cause and consequence. Before him who has acquired the mental poise necessary to contemplate impartially the relations between conditions and conclusions, the affairs of the moment and of all eternity are spread out. The ability so to evaluate life is human and divine and is not experienced within the limitations of mere animal existence, though the struggle for daily bread is shared with the animals.

Mathematics is more than a mundane tool, more than a state of mind, more than a process of nature, because it can erase time, because it can project a human mind beyond the limitations of the attendant circumstances of the moment. This is obvious to him who sees mathematics as "the science of necessary conclusions"; for him who sees mathematics only in terms of numbers the entire meaning of mathematics is obscured.

Responsibility of the learner. Reference has already been made to the fact that acceptance of the existence of a relation of multiplication between the symbols x and y, when written in the form xy, calls only for the passive agreement of the learner to abide by a convention for which he assumes no responsibility whatever. It is at this point that the responsibility of the learner begins. It is also at this point that the teacher has an opportunity to indicate his grasp of the idea that algebra is a forceful instrument for the exercise and stimulation of thinking, instead of a

congeries of facts and processes. If x and y are any two numbers and xy is their product there follows inevitably in fact, but not inevitably in point of psychological activity, the conclusion that if a and b are two numbers, then also their product is a certain quantity expressed by ab. It follows, likewise, that if p has the value 3 and q has the value 7, the value of pq is a certain definite number, 21. The pupil who has been deprived of the opportunity to reach by his own effort the conclusion inherent in the symbol xy, after its primary significance has been explained to him, has missed forever the greatest value that inheres in that particular kind of a situation. He likewise has missed the thrill that comes from a sense of power in being able to reach, for himself, a definite goal which can be attained only by the exercise of an ability which is in the highest sense human. To give a pupil a set of exercises in which he is allowed to obtain the results, when the numbers in xy assume various values, is desirable in order to establish facility in manipulation. Such exercises, however, should not be considered as supplying the primary needs to which algebra is prepared to minister.

There are many instances in which meanings grow out of repeated associations with the same situation. This is notably true of arithmetic in cases where habit formation is involved; but algebra is concerned with states of mind in which the chief consideration resides in ability to reach conclusions in situations where habit cannot operate, for the reason that the situations of algebra are too numerous to permit of ever being placed upon an automatic basis. Such a basis, moreover, would be undesirable, because the ability to generalize is the very quality which economy of thought energy demands and which algebra is prepared to develop.

Such an expression as $2a + b$ likewise affords an opportunity to introduce an element of learning not contained in a response previously associated with a stimulus, although nothing more than such a response is involved if the pupil is shown how to *substitute, multiply,* or *add.* To attempt to show the pupil the meaning of $2a + b$ by substitution of various values for a and b and the performance of the other algebraic operations is to overlook the fact that the expression leads to a certain conclusion because of its definitional significance, and not because it affords an opportunity for exercise in the performance of certain algebraic operations.

The pupil who *substitutes, multiplies,* and *adds* in a thousand such exercises has not necessarily done more than follow directions given by someone else. Rare as is the ability to establish for oneself the associations between a definition and the manipulative conclusions which follow from acceptance of the definition, algebra is designed to further another and different ability—that of self-direction. The pupil who performs a series of exercises like the above because of a conviction in regard to their meaning that is the result of his own interpretation of the definitional meanings of the expressions is operating on a very different plane from the pupil who merely follows a prescribed plan.

The teacher has no more necessary, and probably no more difficult, mission to perform in algebra than to distinguish between those situations in which convention rules and those in which virtue resides in the pupil's own reaction when he is offered an opportunity to reach a conclusion in the presence of all the attendant conditions. Recognition of, and obedience to, conventions may be—indeed, probably always is—reducible to a stimulus-response basis, but thinking is something more than responding to a stimulus. Thinking introduces an element wholly distinct from any stimulus; namely, the element of relationship between things and ideas. The stimulus serves only to bring to attention or to suggest the items concerning which thinking is done. It is worse than useless in algebra to attempt to "sloganize" or to realize the purpose of the subject through the selection and use of "key words." Even though slogans and key words could be found, they would operate only with respect to those conventions of algebra which require, not thinking, but acquiescence. It is doubtful, though, if slogans or key words could be found which would be effective outside the controlled conditions of the schoolroom. For instance, the statement, "I have just read in the paper that Pocahontas coal is $8.50 a ton; that is $1.00 less than I paid last year," calls for the operation of addition, if the listener wishes to determine what was paid the previous year. Obviously the word *less* cannot be associated with the operation of *subtraction.* The word *times* does not always denote multiplication, though it frequently does. Farmer Wheeler remarks that his cherry crop this year was the best he ever raised, that he sold twenty-one tons of cherries, which was three times the size of last year's crop. One does not *multiply* to obtain the size of the crop of the previous year; he divides.

To employ key words makes the mathematical operation the principal thing when, as a matter of fact, the operation is only an incidental aid toward a proper evaluation of a situation which thinking, not manipulation, has created and organized. The solution of verbal problems is notoriously difficult for pupils, for the reason that difficulties of operation are compounded by difficulties of *which operation to use.* It does not seem to suffice to reduce the problem to simplicity in terms of information with which the pupil is thoroughly acquainted, and why should it? It is not reasonable to expect that operations and forms which have been learned with no reference whatever to their thought qualities should suddenly endow the pupil with properties of thinking, merely because they are to be applied in situations where nothing but thinking will direct their applications.

Fundamental differences between arithmetic and algebra. To the beginner the fundamental operations are ways of doing things and *something more:* they are ways of doing things in order to realize a definite purpose, otherwise they become little except pure memory exercises. Addition in algebra is an illustration of this principle. The rule: "If the signs are alike add and prefix the common sign: if the signs are unlike subtract and prefix the sign of the numerically greater," arrives at a destination purely by means of a mechanical procedure. The rule is sound, but to the pupil it offers not the slightest vestige of an opportunity for self-direction, although this is a place where the results of any type of activity other than thinking seem to produce little effect upon the learner's sense of the reality of his performance. In arithmetic the operation and the facts of addition are reinforced by the ordinary, everyday experiences of everyone. Addition applies to things and conclusions to which the senses give heed. The results of arithmetical addition are due mostly to observation and slightly to contemplation. In algebraic addition none of these statements is true. Algebraic addition pertains to conditions which are as real as those to which arithmetical addition applies, but algebraic addition has nothing to do with anything which can be perceived by the senses. Algebraic addition applies to *qualities* or general ratios which are properties of sensory objects—not the objects themselves, such as time, temperature, or the financial status of a man. The results of algebraic addition are due mostly to contemplation, and observation plays a minor rôle.

In arithmetic, acquaintance with the facts and the nature of addition properly begins with concrete things, and abstraction comes as the result of experience with many exercises involving many different sorts of things. In algebra, acquaintance with the nature of addition *begins with a point of view,* and no amount of experience with exercises or illustrations is of much avail in developing a sense of meanings. An illustration of this is to be found in a textbook in algebra, which appeared a few years ago and has been quite widely used. The authors make a statement to the effect that algebra makes use of the positive numbers of arithmetic and includes negative numbers besides. Here is an instance where a person knew enough about *exercises* in algebra to succeed in writing a textbook without having obtained any understanding of the meaning of algebraic or, as better designated, directed numbers.

As a matter of fact, the positive numbers of the directed number scale are distinct and different from the numbers of arithmetic. It happens that positive directed numbers *look* like arithmetical numbers, but the resemblance is superficial. Every item of manipulative performance with directed numbers can be accomplished by following the proper rule, though he who operates upon this basis works always in the shadow of a deep mystery. His accomplishments represent "a low form of cunning." His mind is intent upon following a rule. Thinking, which is the *sine qua non* of algebra, is not required of the manipulator to any great extent.

Contrast the algebraic mechanician, taught by the precepts of such an outlook as that represented in the textbook to which reference has been made, with the estate of a pupil who has been introduced to directed numbers as a form of measurement devised to accommodate ideas to which arithmetical numbers in no sense apply. In the first place, the quantity of the thing to which the numbers of arithmetic apply begins with zero. Not so with directed numbers which apply to things whose beginning is nonexistent or indefinite. Again, in arithmetic a *zero* measure indicates *not any,* while with directed numbers a zero measure always indicates *some* and never *not any.* In the directed number scale, zero measures more than -17, and a definite amount more; zero likewise measures less than $+29$, and a definite amount less. The relations of -3 or of $+8$ to -17 and $+29$ are precisely of the same kind as the relations of zero to these numbers. The expression "less than nothing," so frequently heard in teaching algebra, is an empty

phrase, having no meaning whatever, either in algebra or elsewhere. When and if the mind of man ever conceives of a quantity less than nothing, that idea will call for a scheme of measurement quite different from any yet devised, just as ideas of time and pressure and potential called for a scheme of measurement with which arithmetic was wholly unable to cope.

No one should be deceived by words, of the making of which there is no end. It is easy to speak of a man in debt as "worth less than nothing." It is also easy to speak of "a hen and a half laying an egg and a half." The active "hen and a half" is quite apt to be recognized as a pleasant figment and so harms nobody. Not so with the less-than-nothing idea, which is often accompanied by the statement that the amount the man owes is represented by a negative number. The confusion of thought arises from not differentiating sharply between the zero of the arithmetical scale and the zero of the algebraic scale of directed numbers. Zero of the arithmetical scale is precisely synonymous with *nothing*, while zero of the scale of directed numbers never, under any circumstances, means *nothing* and inevitably means *something* of the quantity that is being measured.

Referring again to "the man in debt," the amount he owes is not negative nor is the amount he has in the bank or that someone else owes him positive. The measure of the debit and the credit has no quality whatever, nor is there any demand or use for directed numbers in dealing with these quantities. The fact that a number is subtracted does not make it negative, nor does the fact that a number is added make it positive. In the illustrations cited directed numbers serve only the function of *measuring the financial status of the man.* He always has a financial status and his relative position in the financial scale can be denoted always upon the scale of directed numbers, but his relative position cannot be denoted by the scale of arithmetical numbers. Directed numbers are designed to measure magnitudes whose quantity exists within the range of contemplation, regardless of the amount by which any given quantity of the magnitude may be diminished. Wherever a measurable magnitude can conceivably be diminished down to zero (nothing), there arithmetical numbers *only* can be used to denote a given quantity of the magnitude. Where a magnitude can conceivably be diminished indefinitely, but can never be diminished to a zero quantity, there only directed numbers apply

and where "zero" is used to denote a measure of such a magnitude, the word means something totally different from "not any"; it means a relative quantity which is greater, or less, as the case may be, than some other measure of the quantity. The expression "absolute value" is correct and acceptable unless coupled with the phrase "without sign," as though -7 or $+4$ or ± 1 could be changed to absolute numbers by removing the signs. A sign cannot be taken away from a directed number, nor can a sign be given to an arithmetical number as an expression of measure.

One of the effects of the fundamental differences in the nature of directed and arithmetical numbers is to give different meanings to fundamental operations in the two systems. Referring again to addition as an illustration, it may be noted that the sign "+" in arithmetic has only one meaning: namely, that of the imperative verb, "add." The sign has this same meaning in algebra and two more besides, making *three* in all. In algebra the sign "+" indicates, besides the operation of addition, *direction* and *position,* and *no two of these three meanings ever coincide.* A good exercise in checking one's understanding of directed numbers would be the adoption of distinct symbols for the three meanings of the sign "+." When a person can employ the three separate "addition" symbols with certainty and clarity he is well along toward understanding the nature of directed numbers and of the operation of addition. One may be a perfect operator without the ability cited, but without it the so-called algebraist is a mere mechanical manipulator; he understands the fundamental nature of algebra no more than does the domestic sheep understand the laws of mechanics, although he may operate to perfection a treadmill for developing power to churn butter.

In arithmetic a pupil always has within reach some tangible material by which he may check both the accuracy of his ideas and the accuracy of his manipulation in connection with addition. No such individual safeguard is to be found in the addition of directed numbers because they deal with, and apply to, intangible things exclusively, where the word *intangible* is used in the sense of being imperceptible to the senses. This does not in any manner imply that operations with directed numbers cannot be checked and checked conclusively by the operator. The checks in arithmetic and in algebra are of a different nature. This suggests that the approach and the learning process in the two subjects should be

different; at least they must be different if that which is learned in the two subjects is to be consistent with the checks.

One of the things upon which we modern educators especially felicitate ourselves is the extent to which definitions have been banished from the schoolroom, and it is a fact that much improvement has been wrought in education by allowing meanings to grow out of experience. That plan of approach works with ideas that are, or can be made to be, the outgrowth of an individual's background. As guides to conduct of a mathematical nature there are intuition, experience, a stated rule, or a stated definition. Algebra does not make use of the first and second in the same way as does arithmetic, if at all. For instance, addition always signifies *more* in arithmetic, but either *more* or *less* in algebra. Algebra, being the result of abstraction, generalization, and reflection that apply to ideas wholly beyond the reach of arithmetic, is dependent upon rules and definitions, especially the latter, for its very existence. Whether in arithmetic or algebra, that which constitutes any operation depends upon a definition. It does not follow that equally in the two subjects an individual's comprehension grows out of the direct application of a definition. In algebra it happens that, by rule, a person may, and frequently does, operate indefinitely without obtaining a comprehensive idea of what the procedure is about. The reason seems to be that such manipulative procedure is based upon the laws of learning that apply to a realm of things rather than to a realm of ideas. Child study alone is well enough for the teaching of childish subjects, but algebra is not a subject for a childish mind. The laws of learning and of instruction depend alike upon the subject and the pupil. The hackneyed phrase of the teachers' institute, "I teach pupils, not subjects," continues to win applause, but it is the expression of educational sophistry. Even a sheep herder can justify his existence only by knowing better than the sheep where green pastures and safety lie.

In teaching, the questions *what to do, when to do it,* and *how to do it* imply an array of choices not only in subject matter but in methods. A course of study can no more be based upon a study of the child alone than upon a study of the subject alone, though for the most part educational reformers of this day have often given comprehensive knowledge of subject matter a rather wide berth. So long as civilized life and living endure, society is going

to be obliged to depend upon, and to be guided by, its social heritage —otherwise there is no civilization. Schools cannot afford to take their educational cues from the mind of the child, any more than they can afford to be dominated by considerations of the subject matter alone. The naïve assumption of some elementary educators that all subject matter into which the child would not automatically evolve himself should be discarded, may appear to be a more humane philosophy than that of the old-fashioned subject-matter-dominated curriculum, but in reality it is less well considered. Education is a matter of administering not only to the child's needs as a child, but to the child's needs as a member of society as well Education is a matter of adjustment to customs and ways of thinking and levels of attainment with whose creation the child himself has had nothing to do, and at which he never would arrive by himself. It would seem as though frank recognition should be given to these considerations in shaping both the content and the methods of the curriculum. Laws of learning may never change, but the amount and quality of learning change with each generation, and with the increasing age of every individual. These facts affect the topics of mathematics and the *methods of presenting these topics* that are appropriate to the age of the pupil

Algebra belongs to the degree of maturity attained by a pupil who has reached adolescent years—the age of reflective thinking. He learns algebra in a different way and with a somewhat different purpose from that used in learning arithmetic. One of the most striking illustrations of these differences is to be found in the parts that definitions play in the two subjects. In arithmetic the definition is subordinated, while in algebra the definition is often the point of departure from which learning proceeds This applies with striking force to the subject of algebraic addition. It appears as though the only way in which meaning can be given to algebraic addition (and this operation is used by way of illustration) is through the medium of the definition. Algebraic addition means to effect a change from a fixed measure by a certain amount in a given direction. This makes the addition of $+8$ and -5 a *totally* different operation from that of the addition of the unsigned numbers *8* and *5* of arithmetic. In the latter case the quantities represented by the figures *8* and *5* may be shoveled into a box in any fashion one pleases; then if one looks in the box he will find a quantity represented by *13*. The symbols *8, 5, 13* represent identi-

cally the same things, and the sign "+," if it is used, has only one meaning. But in the addition of +8 and −5 the case is quite different. In the first place, the quantities represented by these figures may not be "shoveled into a box" because they do not apply to things that can be handled in that way; so, from the nature of the case, the pupil is forever barred from getting the meaning of the algebraic operation in the way in which he can acquire a meaning for the arithmetical operation. In the second place, the symbols +8 and −5, when the two numbers are to be added, do not represent identically the same things. One of these numbers assumes its quality of definiteness because it belongs to a definite place in the directed number scale of measurement, and not because it represents a definite amount of the characteristic that is being measured. The other number represents a definite amount of the characteristic that is being measured and has nothing to do with the directed number scale. The operation, then, of adding two numbers in algebra is a totally different operation from that which goes by the same name in arithmetic, and to reduce the operation, by means of rules, to terms of arithmetical addition (and subtraction) wholly conceals the real nature of the operation. Moreover the *signs* of the two numbers +8 and −5 do not represent the same thing, one being a sign of *direction* and the other a sign of *position*. As to the result, it makes no difference which of these numbers, and consequently which of these signs, is taken as an indication of position, but they never represent the same thing. Suppose that this office of position is assumed by the number +8. According to the definition of addition, the process of adding −5 to the number +8 requires a movement from +8 of five units in a negative direction along the directed number scale. The point to which this takes us on the number scale is the *sum* of the two numbers. The addition process, it will be noted, is not, as in arithmetic, one of combining two measures of the same kind into a single measure of the same kind as both of the original numbers represent. The result is, of course, +3.

The command to "add −5 and +8" contains no intimation of which number represents the starting point. Suppose −5 were taken as this point. The addition of +8 requires a movement of 8 units in a positive direction along the directed number scale. As before, the point reached on the scale is +3 and this represents the sum of the two numbers.

The algebraic operation of subtraction offers in measuring and in the nature of the process the same striking departures from arithmetic that are to be found in addition. In arithmetic, so far as the possibility or the validity of thinking is concerned, it is absolutely immaterial whether we think of the difference between two numbers as measuring "what is left when one is taken away from the other" or meaning "what must be added to one to produce the other." In algebra, on the other hand, the "take away" idea is unthinkable, because the subtrahend may be larger than the minuend. There is no choice of methods in algebra; one is compelled by the nature of the case to think of subtraction as "the process of finding *what number must be added to a certain number* to produce a certain other number." Another interesting contrast between arithmetical subtraction and algebraic subtraction is found in the use of the word *difference*, which is perfectly proper in arithmetic and has no meaning at all when used without qualification in algebra. For instance, the answer to the question, "What is the difference between 6 and 11?" is unique and therefore clear. The question, "What is the difference between +6 and +11?" has no meaning, because between the numbers +6 and +11 there are differences, but not *a* difference. In arithmetic it is unnecessary, in speaking of the subtraction of two numbers, to specify which is *subtrahend* and which *minuend*, since there can never be any ambiguity; in algebra it is imperative that numbers constituting the rôle of subtrahend and minuend, respectively, shall be specified. The signs of the numbers constituting the subtrahend and the minuend are each signs of position, while the sign of the result is that of direction. The word *difference*, it should be noted, is correct as expressing the relation between the two numbers, the result of subtraction, when the office of the numbers has been specified, but not otherwise.

The inability of the "take away" idea to function in algebra may have some bearing on the controversy over teaching the Austrian method of subtraction in arithmetic. A scientific aim would seem to point clearly to the Austrian method as having the advantage, since it is the only one which will provide a point of view that can carry over from arithmetic to algebra.

Use of the equation. The contrast is no more pronounced between meanings associated with fundamental operations in arithmetic and algebra than is the distinction in the use of the equation in the two subjects. In arithmetic the equation is a convenient

form of a statement of fact pertaining to a particular situation. In the equation,

$$12 = 4 \times 3,$$

or in this equation,

The cost of 2 quarts of milk @ 11¢ = 22¢,

there are expressed the important relations of equality, but no other relationships between the numbers or the ideas. One member of each of these equations (the subject member) fixes completely a form that the second member may take. The mathematical element of a necessary conclusion is present in each case in the form of a definite number. There are no further implications; were the sign for equality or the word "equality" to be omitted altogether, the relationship between the two members of each equation would not be affected, either in fact or with reference to the character of the relationship, which is one of equality.

Now compare the algebraic equation with that of arithmetic and for illustration take a very simple form, as

$$x + 2 = 9,$$

or a slightly more involved equation, like

$$p = 2w + 6.$$

In these equations there is expressed, as in arithmetic, the relations of equality, but there the resemblance ceases. Neither member of these equations determines in any degree whatever the form which the second member may take. Were the symbol for equality to be omitted there would remain no relationship, or implication of relationship, between the members of these equations. Evidently the sign, or the fact, of equality is here a circumstance quite different from that encountered in arithmetic. Since that which constitutes the mathematical element in the arithmetical equation, namely, a definite number or form which must attach to the statement of one member of the equation, is lacking in algebra, the element of necessary conclusion must in algebra, as contrasted with arithmetic, reside not merely in the inclusion in the equation of something more but also in the inclusion of something different. Those elements of difference which enter the algebraic equation in the way of excess over and distinction from the arithmetic equation are characterized by the words *general* and *function*. As to meanings these words have nothing in common, but they are always

associated, and neither idea can exist without the other. Failure to be able consciously to associate the ideas of generality and functionality simply means that neither idea is understood and that the significance of the algebraic equation is uncomprehended at best.

As an aid to thinking the equation ranks among the great inventions; its function is not primarily to convey information, but rather to establish a logical relation between quantities or ideas whose association is subject to law. It is perhaps not an anomaly, but it is at least a puzzling psychological phenomenon, that the equation can be treated indefinitely as a machine without producing upon the operator much of an impression regarding the chief purpose which the equation is designed to serve; namely, an aid to thought processes. This fact is attested by an abundance of statistical material, as well as by the difficulty that is commonly encountered in being able to employ the equation in arriving at the conclusions implied by verbal problems. All that any problem ever does is to state some facts which challenge one's ability to determine whether those facts necessarily predicate some conclusion. Usually the attendant circumstances are such as to indicate the general nature of the conclusion that is desired, but not its exact form. The general nature of the conclusion sought may even dictate what facts shall be assembled to enable one to reach the conclusion. Thus the time that it will take to drive from Detroit to St. Louis is determined by looking up the distance on the road map, consulting one's experience relative to speed, and then using these data to reach a conclusion regarding time. If the data assembled and their treatment answer the question expeditiously and certainly, that would seem to constitute a rather final indication that all of the processes which have been employed are perfect.

The interpretation of that which establishes perfect treatment of the equation

$$x + 2 = 9$$

must necessarily depend upon the nature of the conclusion that is desired. Suppose the question is, "What value of x satisfies the equation?" A method that reaches the result with a celerity, ease, and neatness that have never been equalled is expressed by the transposition dictum, "Put the 2 into the other member of the equation, change its sign, and unite terms."

Another method by which the question may be answered is the

employment of the axiomatic treatment embodied in the direction, "Subtract 2 from each member of the equation."

Each of these methods, the transposition and the axiomatic, introduce, by implication at least, the idea of generality, inasmuch as the question itself carries with it the implied observation that x is a number to which there attaches variability or generality of value. These methods completely leave out of consideration the idea of functionality. Solving a problem by transposition or the use of the axioms probably involves not even a vestige of appreciation of the reasoning which the equation is designed to further.

A screen door may be erected to keep out flies; it may be kept closed by springs. A dog may easily be trained to open the door, and when so trained he is an educated dog, as compared with another dog who cannot open the door. However, though educated, the dog with the ability to open the door may exercise the ability a thousand times or more, and after the thousandth experience he will have no knowledge whatever of the purpose which the door is erected to serve, nor will he know whether it is a spring or a tack or the moon that holds the door closed. The dog is a perfect operator. He is educated without knowing anything. He manipulates the door because it is there and he wishes to get past it, but for no other reason. It is quite conceivable that a person as unacquainted as the dog with the laws of bacteriology and of sanitation might pity the dog's efforts and prop the door open, or, if the door has to be closed he might provide another unimpeded entrance. Such a "humanitarian" act would not be unintelligent, though it would betoken ignorance. Moreover the act of eliminating the door would be entirely consistent with all the information that could ever be obtained by observing the dog. No study of a dog's anatomy, or of his instincts, or of his canine nature generally is ever going to justify the existence of that screen door, though upon the door may depend the health, the happiness, or the very lives of children and adults.

Algebra is not a screen door, because the latter is a transitory device, while algebra contains the elements of eternal principles, but both are alike in that they minister impartially to the informed and the uninformed. Both must be kept in their places through the influence of intelligence that is educated beyond the limits of observation of the things themselves. If questionnaires were to be circulated among the population at large, it would be difficult to

find people who could point with convincing certainty to specific instances where either algebra or screen doors entered with determining effect into their lives. It is reasonably certain that the reactions to the questionnaire of those whose knowledge comprehended fundamental principles would differ from those who had concentrated upon observation, however intense, of either the operators or of the things themselves.

Algebra shares with other human inventions and discoveries the common requirement that its evolution depend upon its purpose and not upon its superficial aspect. The ability to make comparisons in terms of definite precision has been gained slowly and is still confined to a relatively narrow range of things, but it is an ability that is so essential that we could not conceive of civilization without it. This ability to formulate comparisons includes, almost inevitably, a body of mechanical forms and conventions. That is true of the equation, which is a machine *and much more,* but wherever the machine is found there is a tendency to "make the wheels go round" regardless of the purpose which their turning is designed to serve. The equation is such a versatile machine that manipulation can easily be made a varied and interesting exercise, which even takes on the appearance of respectability. Whenever a habit is to be set up, mechanical practice is justifiable and essential, but the equation fits into a scheme of things where its solution is an incident to a more significant situation which gives rise to the equation. This is the case generally in mathematics. The great difference between arithmetic and algebra in this particular resides in the nature of the ideas with which the subject deals, the subject of algebra representing a mental activity which deals with properties of things into which comparisons of relative weight enter, not through the avenue of the senses, but come as the result of one's being able to comprehend properties of things which are altogether distinct from the things themselves.

The solution of the equation requires an application of the fundamental operations of algebra, but this is not true of the situations that give rise to the equation. Such relations as those of time, distance, and velocity involve ideas of dependence which are quite as fundamental as are the operations by means of which the conclusions, in any specific instance, may be reached.

There is no such thing as an algebraic equation in which the ideas of *more* and *less* are not included as well as the idea of

equality. It is because of their failure to take *more* and *less* into account that solutions by transposition and axioms are inadequate.

In the equation,

$$x + 2 = 9,$$

the most important question is not "What value of x satisfies the equation?" The important question is, "What relations exist between the elements which compose the equation?" That is precisely the question that is back of every algebraic equation that has ever been formulated for the purpose of answering a genuine requirement outside the mere solution of an equation after it has been formulated.

To ask of the equation cited, "What is the relation of the second member to x?" or to ask, "What is the relation of the second member to 2?" puts into the equation the necessity for exercising an ability in which manipulation plays a secondary rôle, and in which thinking plays the kind of part which that activity always must take in any situation where the equation needs to be employed. The treadmill of manipulation is present always where one seeks a conclusion in terms of precision as well as of logic, but there is comparatively little in common between him who calculates by rule and him who proceeds with some degree of appreciation of the purpose which the machinery serves. Algebra is primarily a method of thinking.

THE FUNCTION CONCEPT IN ELEMENTARY ALGEBRA

By N. J. LENNES
State University, Missoula, Montana

A discussion of the kind exemplified in this chapter is necessarily colored by the predilections or "bias" of the writer. A purely objective or "scientific" treatment, such as is possible in developing a mathematical theory or in reporting laboratory experiments, is clearly out of the question. This is of course obvious to the reader, but this statement will remind him that the writer is also aware of this fact and hence will not take differences of opinion too seriously. All that the writer claims is that since the first years of the century he has kept in fairly close contact with opinions and practices in the teaching of algebra, and has made his best efforts to appraise the value of innovations that have been made or proposed.

Variety of opinions. During the last thirty years there certainly has been no lack of interest in the subject of the teaching of algebra. It has been dealt with in a veritable flood of articles, monographs, books, and reports. My notes contain excerpts from, or remarks upon, more than fifteen hundred of these dealing directly or by obvious implications with this subject. These are expressions of those whose ideas have reached the printed page. Unreported addresses together with fugitive printed articles that have escaped me are no doubt very large in number and there have been endless private and informal discussions. The most outstanding fact about this vast outpouring is that something is supposed to be wrong with the teaching of algebra and that a real remedy is believed to be possible.

But while there has been fairly general agreement that the patient is ill, some think the ailment is not very serious, while others think him near the grave. There has naturally been, therefore, great divergence in the remedies proposed. This divergence in opinion as to the seriousness of the ailment and the needed remedies is no doubt due to a more fundamental cause than is

usually recognized. Some are occupied with changes that they regard as immediately possible without serious disturbance of the more general state of affairs. They think of the course in algebra as a part of a whole over which they have little control. They presuppose the teaching in the grades below the high school as it now is, and also the objectives to be attained as controlled by the present requirements. These people are fairly closely limited in their speculations and in their experiments; the changes that appear feasible to them are not very great or fundamental. There are others who attempt to go back to first principles to think out anew a philosophy of education. Not infrequently these people find the present very little to their liking; they would like to "shatter it to bits, and then remold it nearer to the heart's desire." In their thinking about a possible course in algebra they are gloriously free to wander as fancy dictates. Between these extremes there are all shades of opinions and proposed plans.

Scientific evaluation of educational methods. This is an age of science: Why not subject educational questions to the methods of science? Why not settle them once for all, as questions in the sciences have been settled, and have done with this welter of inconclusive speculation? The difficulties are overwhelming. Questions that involve relative valuation of fundamental elements of life cannot be answered in this way. The answers to such questions are now, and must forever remain, mere matters of opinion. Who shall decide which would be better for me, a course in German which would enable me to read Goethe's *Faust* in the original, or a course in business administration which would enable me better to understand the industrial and business world in which I live, or a series of courses in education which presumably would make me a better teacher? I certainly do not know and I can conceive of no test or experiment that would tell me. The fates willed that I should have one of these and not the other two, and I am fairly satisfied, though I very much suspect that a conclave of educators would have ordered my life differently.

But why worry about "scientific" experimentation in connection with this subject? Has anyone tried seriously to put to a crucial scientific test the question of functions or no functions in elementary algebra? Not so far as I know. In dealing with this problem we have been obliged to get along, blundering no doubt, as the human race has been getting along, meeting millions of emergencies where

scientific solutions have not been available. But no one supposes that intelligence has been without value in these emergencies. Common-sense conclusions and well-considered opinions, though often wrong, have been of supreme value. It is only because they were of such value that intelligence could survive and develop. For this reason a discussion such as the present may be of real value, though we can never be quite sure that we are right.

Definition of function. The sharply formulated definition of function as given in higher mathematics does not really concern us here and even an approach to it will be deferred to the end of this section. Let us begin by considering a simple example. Very early, probably in the third grade, one learns that 4 quarts make one gallon and that to find the number of quarts we multiply the number of gallons by 4. This relation between the number of quarts and the number of gallons is a functional relationship.

This is quite apparent when we represent the number of quarts by q and the number of gallons by g and write $q = 4g$. However, we do not talk about the study of functional relationships in the third grade. Why not? The reason is this: The rule which we used in the third grade, and the equivalent equation which we use later, have a functional aspect, but we pay no attention to this aspect when we are learning the multiplication table, nor do we necessarily pay any attention to this aspect when we use the formula $q = 4g$. Indeed it is precisely the neglect of this aspect that is complained of by those who desire to infuse the study of functionality into algebra. What is this functional aspect which has been so sadly neglected? It is simply this—as g changes, the function $4g$ (or q) changes in a certain way (namely, increases four times as fast as q). The study of the function is the study of its behavior, or the "march of the function," as Professor E. H. Moore used to say.

Every rule of arithmetic and every formula represents a functional relation, but the use of the rule or the formula need not involve any consideration of its functional aspect. Suppose we are using a table of squares in practical computation. The table may be represented by x^2, which is of course a function of x. However, when using the table the squares of particular numbers are found without any special regard to the functional aspect of x^2. It is only when the variation of x^2 as related to the variation of x is brought under consideration that we come to study the

functional aspect of the facts in the table. The derivatives of a mathematical expression, though these do not even indicate the value of the function for any value of the variable, represent its functional aspects in a very definite way.

The following is the usual definition of a mathematical function of a single variable: *Any mathematical expression, containing a variable x, that has a definite value when a number is substituted for x is a function of x.*

In practice the word "function" is now being used in a much more general sense. Thus economists say that the price of wheat is a function of the amount produced, and, of course, of many other variables; and criminologists are asserting that the amount of crime is a function of unemployment. For the purpose of secondary mathematics we may say that a *function is a quantity which varies in a definite way as some quantity involved in it varies.*

Representation of functions. A function may be represented, or defined, by a verbal statement such as we use in arithmetic, by an equation or formula, by a table, or by a graph. Thus we may say that the circumference of a circle is obtained by multiplying its radius by 2π, or we may write $C = 2\pi r$, or we may construct a table giving the circumference for a large number of values of the radius, or we may construct a graph of the equation $C = 2\pi r$. Of these representations the graph is most certain to bring out the functional aspect to the beginner. It soon becomes evident that the graph represents the function only to an approximate degree of accuracy, and that the table represents it only for a selected set of values of the variable. The formula and the verbal statement represent the function completely, and provide a rule by which values may be obtained to any degree of accuracy. When irrational values are involved the table is of course only approximate.

Importance of the function concept. The world in which we live is constantly changing. While we regard some of its elements, at least provisionally, as fixed or constant, we are forever occupying our minds with its changing or variable aspects. This aspect of variability in the world has come to engross our attention increasingly as a result of scientific investigations of the last hundred years. Moreover, we think of these variable elements as interdependent; a change in one of them causes, or is accompanied by, changes in the others. That is, we regard the universe, in a way, as a huge equation containing a vast number of variables. In many

cases it is possible to show that a particular variable is but slightly affected by any except a small number of these variables, possibly only one of them, and it has been possible also to formulate the law of interdependence of some such variable. In this way we have been able to represent natural phenomena by means of mathematical formulas or functions. The question which we are constantly raising is: How is this or that element in our environment (or in ourselves) changing? The equivalent question in mathematics is: How is this or that function behaving? How is the function "marching"?

It is important not only that we learn to know how certain functions behave and thus understand the character of important changes in the world in which we live, but also that we become accustomed to think functionally, that we learn to think not only of static, isolated facts, but of a set of facts as constituting a moving, changing sweep of things. To what extent does a change in the independent variable cause a change in the function? Is the change in the function great or small when compared with a change in this variable? Granted that one phenomenon depends upon another—say, for example, that the rate of yield of a potato field depends upon the amount of fertilizer used—what is the character of this dependence? To consider potatoes and fertilizer, to what extent is the yield increased by increase of fertilizer? What is the effect of the first dollar's worth of fertilizer per acre? of the second? of the third? and so on. What is the amount of fertilizer, an increase beyond which does not increase the yield? It is clear that absolutely definite answers to these questions are impossible since many other elements are involved, but it is also clear that approximate answers of real practical value may be obtained for fairly definitely known conditions. The point is that we need to become accustomed to think of the quantitative aspect of such dependences, if you will of the *rate of change,* or the derivative, of such functions. A fertile field for error is found in practical reasoning in dealing with rates of change. Granted that increased advertising will increase sale, *how much* will sales be increased by a given additional expenditure for advertising? That is the crucial question, and we very much need to become accustomed to consider the "how much" in such cases.

But we are still on doubtful ground. Will a study of the character of the variation of simple functions expressed in mathematical symbols lead us to consider the variation of indefinite but press-

ingly practical functions such as those we have just mentioned? The hard and fast believers in "non-transfer" of the results of learning will answer with a confident "No." Very probably the truth is that practice in thinking of the variation of mathematical functions does not carry over automatically, but that attention once being attracted to the more general kinds of function and their variation, practice with mathematical functions will make one take more readily to the other kind. The very idea of functional dependence and of types of variation will be of real help. If we are not permitted to believe in this kind of "carry-over" we shall be driven to the conclusion that valuable education beyond the most directly and immediately practical is impossible.

The value of the function concept within the domain of mathematics itself is pretty well agreed upon. The remark attributed to Klein that the function concept is the very soul of mathematics has been quoted extensively and with very general approval. In the latter part of this chapter where the question of how functionality may be studied in elementary algebra is considered, this phase of the subject will be treated in more detail.

Early opinions about the function concepts in algebra. It will be remembered by those acquainted with the literature of this subject that in the first years of this century it was proposed to make functionality the unifying element in beginning algebra. This proposal appeared in print and was discussed widely and enthusiastically, pro and con, at teachers' meetings. Reports were made of classes conducted according to this general principle and at times I, for one, was on the point of being completely converted. One difficulty was, so it was said, that there were no texts suited to this purpose, and a little later it was complained that publishers were too short-sighted and too "mercenary" to publish the sort of books needed. Just what led to this conclusion I can understand more readily now than I did then. As I understand the literature dealing with the teaching of algebra, this extreme view has gained no great headway even among those who contribute to the periodicals or who write books on teaching, and no doubt these are more "advanced," and even more radical, than the rank and file of those who teach.

Use of the word "function" in recent practice. I know of no method of obtaining a correct judgment of the way a subject is taught that is anywhere nearly as effective as a study of the texts

that are used. The great majority of teachers follow the text very closely, for what they consider good reasons. They are exceedingly busy and have little time to formulate an independent treatment even if they had a desire to do so. Moreover, many teachers do not wish to depart from the treatment of the text, because they feel that in a first course the pupil should not be confused with different presentations—one by the teacher in the classroom and another by the text, which the pupil should be encouraged to read.

To find how our textbooks deal with this matter I examined eight elementary algebras published within the last five years. Some of these are what may be called conservative books and include revisions of very widely used texts of long standing. Others are "radical" or "progressive" (according to the bias of the writer). They include all the newer books that are receiving serious attention. In four of these books the word "function" does not occur. Two of these four are the most widely used texts in the country. A third one, a new book, is written by men who have taken very prominent part in discussions and investigations related to the teaching of mathematics. The fourth is written by teachers of long experience and well-known ability, who have followed the literature on this subject for a quarter of a century.

In each of the remaining four the word "function" does occur. One of these introduces the word early and there is a discussion of functional dependence extending (including questions) over two pages. A little later the word is used in connection with a straight-line graph. Here we learn incidentally that "functional graphs are algebraic"—while graphs representing statistics do not represent functions. The authors of this book evidently belong to that group of teachers who regard knowledge of subject matter of minor importance for the teacher provided he can "teach." There is no recurrence of the word "function" until the last few pages of the book. In another of these texts the word is first used about the middle of the book after which it is promptly dropped. The remaining two of these texts introduce the word in the last pages of the books in connection with a treatment of "variation."

Variation. Continuing our study of these same texts we find that "variation" is considered more extensively than would seem to be indicated by the very limited use of the word "function." All these texts give some treatment of variation, though the extent of this treatment differs greatly. One very widely used text has

a couple of pages in the middle of the book devoted to this subject, and another well-known and also widely used book has a few pages at the end, while one new text devotes one-eighth of the whole book (the last one-eighth) exclusively to different kinds of variation. Another new book has more than thirty pages on variation in the latter part of the volume.

The result we get from a study of these texts is about as follows: In the majority of classes in elementary algebra, possibly sixty-five per cent of them or more, the ideas of function and of variation are introduced in a perfunctory way which cannot leave any lasting impression on the minds of the students. In another group, possibly twenty-five per cent, there is a more intensive study of variation at the end of the year. In no case can functionality or variation be said to be more than one of a large number of topics studied in the course. In the majority of cases these topics have no organic relation with the rest of the work and appear to be brought in for extraneous reasons. What these extraneous reasons may be we can only conjecture. In writing textbooks it is sometimes thought best to put in a little of everything so that all wants may be satisfied. Certain it is that in the vast majority of our classes in algebra, not only have we failed to make the function concept an all-pervading or unifying principle, but the consideration of it is so perfunctory that little would be lost if it were left out entirely. The lip service paid to it on a few isolated pages is of very little consequence.

No doubt a small number of teachers are developing the idea of functionality in a sane way, making it an organic part of the course. One recent text that has come to my notice does this very well, though, in my opinion, it has other characteristics which will prevent its extensive use under present conditions.

The remainder of this chapter will be devoted to a somewhat detailed consideration of how and to what extent the function concept may be made an organic part of a first course in algebra, while this course is also made to satisfy the other requirements now made of it.

Present requirements of a first course in algebra. Whatever may be our personal beliefs as to what should be, we are met at the very outset by the inexorable fact that no proposed course in algebra will be adopted to any considerable extent unless it contains among its main objectives certain aims which may be catalogued

rather easily. Nor must the course contain extraneous matter which will make it difficult for a median group of pupils to attain these objectives. It is often said that there are three essential elements in elementary algebra: namely, the use of letters to represent numbers (this includes the formula), the introduction of signed numbers, and the use of the equation. This is all quite true and also quite general. A little further inquiry brings more detailed requirements to the surface. We may make the following list of requirements:

1. Algebraic notation and the order of indicated operations including the formulation and evaluation of formulas.

2. The four fundamental operations on algebraic expressions. These involve a very considerable amount of factoring.

3. A treatment of signed numbers based upon concrete situations to which these numbers are applicable. This includes the fundamental operations on such numbers.

4. The solution of equations in one unknown in the first degree.

5. The solution of sets of two or three first-degree simultaneous equations.

6. Some work in handling radical expressions. This usually includes something about fractional, zero, and negative exponents.

7. Solution of a large number of verbal problems (story-problems) in which these various topics are used.

8. Solution of the simplest quadratics.

These requirements are as obligatory, whether we regard them as reasonable or not, as is the observance of our most firmly intrenched taboos. In very many classes there are, for instance, some pupils who will take the examinations of the College Entrance Examination Board. A teacher who fails, in a marked degree, in preparing intelligent students for these examinations usually would not last long. In almost every class there are some pupils who will go on with some further study of mathematics. If it should be found that students of high intelligence have had a course in algebra but have had no experience with one or more of the topics mentioned above, inquiry would be made about the school where this course was given and unfavorable conclusions would be drawn. If it should become evident, for instance, that such a student had never had any experience with factoring of expressions of the types $a^2 - b^2$ and $x^2 + 7x + 12$, or with the reduction in $\sqrt{8} = 2\sqrt{2}$ or with the solution of a pair of linear equations, or with the simpli-

fication of expressions containing fractions, his work in elementary algebra would be regarded as below a reasonable standard and this would find its way back to the school and the teacher, and, except in very unusual circumstances, would result in a change that would bring the course more nearly into conformity with accepted standards. The point I am trying to make here is that, whatever changes we may contemplate, they must be such as to leave room for the carrying out of certain well-defined work. Nor must the new elements contemplated in the proposed changes encroach very much upon the time and energy of pupil and teacher. The course is already quite heavily loaded. Those who think that some of the present material may rather easily be displaced in favor of other material that they regard as more valuable simply do not understand the power of the "gods of things as they are." These gods can be circumvented only slowly and usually by indirection.

Individual differences. There is no longer any doubt that in any normal class the pupils differ greatly in their ability for achievement. It is very difficult to formulate a course of study which is simple enough for the poorest students, and the ablest ones can do a great deal more than is contemplated in courses that are fairly well adapted to the ability of the median student. It is my personal opinion that the greatest advance now immediately possible in almost any course of instruction is along the line of providing appropriate work for students of different abilities. It is not enough that more work of the same kind is given; the work must also differ in quality. In our effort to bring the median student up to a certain standard we have, in my opinion, failed to provide certain work which the abler student could easily understand and which to him would possibly be of greater value than all that is included in the regular course aimed at the median student. True, there are certain practical administrative difficulties, but these, I believe, can be overcome. It is my idea that no course of instruction—a course in algebra for instance—is adequate unless it makes proper provision for more and different work by these abler students. It is with this idea in mind that the remaining part of this chapter is written. In this part I cannot do better (although someone else might) than to outline the treatment of functionality used in an experimental course that is now being conducted under my direction.

First introduction of the idea of function. The first objective is to achieve an understanding of the linear function and its varia-

tion. This is the *objective*, but no sane teacher will use this kind of language early in the course. He will know that an objective of this kind cannot be attained by one grand frontal assault. Preparatory work of many kinds, flanking attacks, the attaining of subsidiary objectives, all these must be planned and carried out. Unfortunately there is no uniformity of usage on one important point that is directly involved. Graphical representation of a function probably serves most effectively to exhibit its variation, that is, its really functional aspect. Some teachers feel that the main object at the beginning of the course in algebra is to learn to do something that is characteristic of algebra, namely, to use the formula and the equation. If the Cartesian system of coördinates is also to be introduced at the start, the course will be badly cluttered. We shall be trying to teach too many things in the first few weeks. Others are willing to begin the work with a study of graphical representation. Some texts seek to fit into both plans by giving what is essentially a chapter on graphs under the caption "Introduction," which is followed by a regular Chapter I, usually dealing with formulas. But, whichever plan is followed, functions must eventually be studied directly from the equation or formula and also from the graph. Certain functions that are very important from the practical point of view are best studied from tables.

Study of functional aspect from the equation or formula. Practically all the facts that are needed for the study of the changes of a linear function may be brought out by questions. To face the issue at the beginning by explanation and definition is fatal. Start with the simplest possible function, such as $q = 4g$. Find the values of q for a series of values of g. Then put questions such as: What happens to q when g is increased from 2 to 3? from 3 to 4? from 4 to 5? If g is increased by 1, what happens to q? Bring out that the answer is the same no matter from what value the increase of g is started. Then go through with the same story, only more briefly, when q decreases.

Now ask the question: What happens to q when g increases? Many pupils answer "g increases by 4." These youngsters do not make sharp discriminations. It is finally brought out and the pupils are caused to put the thought into their own language, that q increases 4 times as fast as g. The teacher or the text may now say: Suppose we think of g as changing or varying gradually, then what happens to q? The language may thus be changed so that

"variations" is used instead of "change" and the pupils are brought to say that q varies 4 times as fast as g. At this stage there are no definitions of variable, function, or variation. As a series of formulas are brought in for the main purpose of studying the formulas themselves, similar questions are put for the purpose of learning the character of the functions that are involved. If in the formula $A = bh$, giving the area of a rectangle, a fixed value as 6 is given to the base then the area is found to vary 6 times as fast as the height, and so on. This discussion culminates in a real understanding of a statement such as this: If in $A = bh$ a fixed value is given to b, then A varies b times as fast as h. And also: If in $A = bh$ a fixed value is given to h, then the area varies h times as fast as b.

A similar treatment is now given to the formulas $A = \frac{1}{2} bh$ (area of triangle), $p = br$ (principal, base, and rate), $i = prt$ (interest formula), $c = np$ (cost, price, and number of articles), $c = 2\pi r$ (circumference of circle).

At the end of this chapter when a considerable body of ideas on variation are easily familiar something is said about variation in general and the word "function" is brought in and given an informal definition. Nothing is said at this stage about "direct variation" since this would bring in by implication some other kind of variation. "One war at a time" is the slogan. At this stage the function is represented as a single expression (not as an equation). A number of functions are now given for the purpose of studying their variation. Such are $3x$, $2x + 1$, $5x + 4$, $x - 1$, $2x - 3$. It is pointed out that the function $x - 1$ has (as yet) no meaning when x is less than 1. The utmost care is taken that at no stage is anything mysterious allowed to enter. It is so easy and so everlastingly tempting to fall back on words and empty forms. "When in want of ideas, then a word opportunely steps in," says Mephisto to the student Wagner.

Supplementary work. Along with the main course and closely adjusted to each part of it there is supplementary work which may be undertaken by those "who wish to do so." What this really means is that those who have greater than median ability are given something to do. This supplementary work is on the whole more difficult than that of the main course. Greater originality and ability to take "longer steps" in one's thinking is required. However, this work is so graded that in each section it begins very nearly on the level of the median course and then gradually becomes more

difficult. The last examples or problems in a group can usually be conquered only by the ablest students and then only after persistent and repeated efforts.

The supplementary work to go with that described above is as follows: We consider again $A = bh$, and suppose A fixed. Then if b is multiplied by 2, h is divided by 2, and so on. Definitions are given of direct and inverse variation. The objective is the understanding of the variation of $\frac{k}{x}$ when k is constant and x varies. The means used to this end is a study of the inverse cases of the functions studied in the main course. The last questions are of the type: What happens to $\frac{k}{x}$ as x grows very large? What happens to $\frac{k}{x}$ as x grows very small (toward zero)? Expected answers are: "$\frac{k}{x}$ grows very small"; "$\frac{k}{x}$ grows very large." Other questions are: How large must x be if $\frac{1}{x}$ is less than 0.001? How large must x be if $\frac{k}{x}$ is less than 0.001? These concepts are developed most interestingly in connection with formulas such as $p = br$, and $c = np$, given on page 63.

Linear functions involving negative numbers. After signed numbers have been introduced we consider again the function $x - 1$, which has already been studied for values of x greater than 1. It now develops that this function increases at the same rate as x also when x is negative. Evaluation of the function $3x + 2$ for both positive and negative values of x affords practice in multiplying and adding signed numbers. Again, the rate of variation of the function is found to be the same for positive and negative values of the function. Finally, we inquire what value of x will make this function equal to zero. This leads to the solution of $3x = -2$ and a negative value of x.

Evaluation of functions such as $4x - 7$ and $\frac{2x - 9}{4}$, and the study of their variation, give practice in all the operations on signed numbers. By this means it is also made quite clear that a change from -2 to -1 or from -2 to 0 is a change in the positive direction.

The supplementary work at this point deals with the readings on the Fahrenheit and centigrade thermometers. A drawing of the two thermometers is made on the board showing that 0° and 32°, and 100° and 212° are used to mark corresponding points. Pupils are encouraged to obtain a centigrade thermometer and compare it with the Fahrenheit.

The following questions are then asked in the text:[1]

1. Into how many degrees has the difference between the freezing and the boiling point been divided on the Fahrenheit thermometer?

2. Into how many degrees has the difference between the freezing and the boiling point been divided on the centigrade thermometer?

3. One hundred degrees change on the centigrade thermometer is equal to how many degrees change on the Fahrenheit?

4. If the temperature rises 10° on the centigrade, how many degrees will it rise on the Fahrenheit?

5. If the temperature rises 1° on the centigrade, how many degrees will it rise on the Fahrenheit?

6. What is the Fahrenheit reading for each of the following centigrade readings: 0°, 5°, 10°, 15°, 20°, 25°, 30° . . . up to 100°?

7. If F is used to represent Fahrenheit readings and C to represent centigrade readings, complete the formula $F =$, showing the relationship of these readings.

8. Study the change in F as C changes.

Fractional functions. No topic in functional variation has been found that is directly related to the subjects of factoring and fractions and that is of a degree of difficulty appropriate to the main text. Such topics have, however, been placed in the supplementary work. There it is undertaken to find what happens to a fraction such as $\frac{x+3}{x+5}$ when x becomes very large and also when x is very nearly zero. Next are studied such fractions as $\frac{2x+2}{3x+4}$ and $\frac{5x-3}{2x+1}$ as x becomes very large and also as x is made very small

[1] Note that this problem is optional and is not considered in the class. The pupil gets all his information from the printed page or from his own observation. In our case the teacher has provided herself with the two thermometers and the pupils who wish to work on this problem are permitted to inspect them outside the class hour.

(near zero). Then the same questions are raised for the fraction $\frac{ax+b}{cx+d}$, where all the constants are assumed to be positive.

The last questions in this section are: What is the effect on the value of a proper fraction when the same number (positive) is added to both terms? What is the effect upon the value of an improper fraction when the same number is added to both terms?

Graphical representation of linear functions. After the usual study of the Cartesian system of coördinates, a graph is made of the equation $q = 4g$ studied at the outset. The equation, the graph, and a table from which the graph is made, are placed side by side to show how these all represent the same relation between q and g. Next we construct the graphs of the formula $A = bh$ for $b = 1$, $b = 2$, $b = 3$. These graphs show that the "steepness" of the line indicates the relation between the variation of A and h.

Graphs are now constructed of the function whose variation has been studied earlier; the old questions are recalled and the answers related to the graphs. Additional graphs of the same type (such as $d = rt$, distance, rate, time) are used as exercises.

Next we construct the graph of the equation $x + y = 4$ and for the first time we run across the function $y = 4 - x$ which decreases as x increases. This we call a decreasing function of x. The functions which have been studied earlier are all increasing functions of the variable.

By appropriate degrees this study in the main text is made to culminate in the general conclusion that every equation of the type $y = mx + b$ is represented by a straight line. For this reason $mx + b$ is called a linear function of x. The value of m shows the rate at which the function varies. If m is positive, the function increases and if m is negative the function decreases.

In the supplementary work there is also introduced at this point the graph of $y = \frac{1}{x}$, thus showing graphically the nature of inverse variation.

Graphs are also constructed of the equations $y = \frac{x+2}{x+5}$ and $\frac{x+6}{x+2}$ by first making tables of pairs of values of x and y, then plotting the points and connecting them by a smooth curve. Negative values of x are not considered unless some pupil decides on his

own initiative to do so. We then verify on the graph the conclusions obtained earlier about the variation of these fractions as x varies.

The function $y = \sqrt{x}$. At first it was decided to limit the study of functions in the main course to linear functions. In the supplementary work a two-place table (two decimals) of square roots was to be constructed up to $\sqrt{99}$. Curiously enough nearly all members of each class (two experimental classes are being considered) took part in the construction of this table. For one thing the construction of the table made use of such facts as $\sqrt{8} = 2\sqrt{2}$, $\sqrt{21} = \sqrt{7} \cdot \sqrt{3}$. In the future the construction of this table will be placed in the main course, though some of the more difficult things brought in in this connection will be reserved for the supplementary work.

In using this table, roots of numbers such as $12 \cdot 7$ are found by ordinary interpolation. It is a real "eye opener" to find that in a class of thirty first-year high school students there are a half dozen who will learn to interpolate by reading the printed explanation and, on the whole, have less trouble with it than the median college freshman has. They also use this table for finding approximate roots of larger numbers. Thus the root of 583 is found by interpolating for $\sqrt{5 \cdot 8}$ and multiplying by 10. The construction and use of this table make the pupils familiar with a number of important and interesting mathematical ideas which are of little significance or interest when presented in the usual isolated and unrelated fashion.

A graph representing this table is constructed, and a table of square roots up to $\sqrt{10}$ is constructed on a larger scale. From these graphs the function of $\sqrt{x}$ is studied, the following facts being brought to light.

1. The function is increasing but increases more slowly for larger values of x.

2. The curve is concave downward and hence ordinary interpolation gives values that are a little too small.

3. The curve is more nearly straight for larger values of x and hence the error in interpolation is smaller for such values.

These facts are all verified by actual computation for selected numbers.

The table is also used to construct the curve $y = x^2$. It is pointed out that the second curve is obtained from $y = \sqrt{x}$ by interchanging the rôles of x and y, and that the curves are exactly

the same shape. Some pupils are sure to discover that the second curve may be obtained from the first by folding the paper so as to make the x-axis fall on the y-axis. A page in the supplementary work is devoted to the formula $A = \pi r^2$ (the area of the circle). A table and a graph are constructed by the pupils and the expression, "The area varies as the square of the radius," is used. The work is not laborious since $\pi = \frac{22}{7}$ is used and the table of square roots is read backward to find squares. The formula $d = \frac{1}{2} gt^2$ is used as an additional exercise. The page ends with the statement that the volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$ and that hence the volume of a sphere varies as the cube of the radius. Some pupils undertake to construct a table and a graph for this function.

Inverse squares. In the supplementary work related to the solution of the quadratic there may be introduced the equation $y = \frac{k}{x^2}$. Graphs of this function are constructed for $k = 1$, and $k = 4$. The expression "varies inversely as the square" is used. The law of gravitation is stated and questions of this kind are proposed:

A boy weighs 100 pounds at the earth's surface: How much would he weigh 4,000 miles above the surface? 8,000 miles above the surface? 12,000 miles above the surface?

Trigonometric functions. In the last part of this course, tables of sines, cosines, and tangents are given for the purpose of computing heights and distances. Graphs are made representing these functions, and the rate of variation of each is studied. How fast does the sine vary when the angle is near zero? When it is near 90°? How does the variation change as the angle grows from zero toward 90°? Similar questions are raised about the behavior of $\tan x$ as x goes toward 90°. All this work is confined to angles between 0° and 90°.

The third semester of algebra. For this part of the course detailed description will be omitted. Certain obvious extensions of the study of functions are provided. Tables and graphs of some important practical functions are constructed. In some cases the mathematical processes involved in constructing the table are

beyond the pupils' present attainments, and the required numbers are given in the text, the work of constructing the graph and studying the character of the functions being left to the pupils. An example of such problems is the following: Sales are made on the installment plan whereby 10% is added to the normal cash price. This total is divided into n equal parts, one part being paid in cash and the remainder in $n - 1$ equal monthly payments. Find the actual rate (i) of interest which the buyer pays for the integral values of n from $n = 5$ to $n = 25$. Plot the results, using the values of n as abscissas and the values of i as ordinates and draw a smooth curve through them.

This curve represents i as a function of n. A study of it reveals what the great majority of people never suspect. The pupil who can solve the quadratic will find the values of i for $n = 2$, and $n = 3$, but for higher values he can do nothing with the problem except by methods that involve a prohibitive amount of labor. The facts are therefore stated with the explanation that, in courses on the mathematical theory of investment given in the colleges, methods are found for solving this problem. Curiously enough, there are no tables in existence, so far as I know, which give all the required values of i, though such tables as exist are of assistance in finding them.

In the main body of this course is given also practically all the work on functions which is found in the supplementary work in the first course. For the pupils who studied this material in the first course this is merely a review, and these pupils may spend their extra time on supplementary material for the course they are now studying. There is, further, some work on variation in connection with a study of proportion and its applications.

Summary. In the main course for the first year for which the work on functionality has been outlined, the median student acquires a fair acquaintance with the general idea of functions and of the behavior of the simplest functions. This work is so interwoven with the rest of the course that it becomes an integral part of the whole and not merely something extraneous that may be learned (and forgotten) separately. The extra time required because of this emphasis on functionality is very small and is probably compensated for entirely by the additional light that it throws on other parts of the subject. The more able students who do a large part or all of the supplementary work obtain a considerable understand-

ing of the idea of functions and their behavior. Expressions such as "varies directly as," "varies as the square of" "varies inversely as," "varies inversely as the square of," become familiar to them and are rather fully understood.

The standard course. Since frequent mention has been made in this chapter of a "main" or "standard" course, a more detailed description is justified. After a considerable study of texts and well-known examination papers I have come to the conclusion that it is possible to simplify the course in algebra to a considerable extent and still have it meet the normal requirement now put upon it. This main, or standard, course is a course thus simplified. The whole class is taken through this course in a normal way and a pupil who does the work reasonably well and nothing more gets a grade of *C*, the median grade. There are some other special characteristics of this course, which need not be described here.

The supplementary work and a method of handling it. None of the supplementary work is required of any pupil. The teacher tells the class a straightforward story about the "median" character of the work undertaken in the standard course in the class and states that meeting a minimum requirement is rather good work in this course. "Besides this work there is considerable supplementary work that may be undertaken by those who wish to do so. Practically never is the highest grade given to anyone who does not do any of this work. Before undertaking any of this extra work you should do the standard work well."

Except in some unusual cases the teacher never even suggests to any pupil that he should undertake any of this supplementary work. The purpose is to get him to do it on his own initiative. Now and then an able pupil who is lazy or who does not have faith in himself may be asked why he does not "tackle" some of it. One of the brightest boys I have ever known replied to such a question: "Do you think I could?" "Just try it and see." The result was that he did all the supplementary work and by the end of the year was well on his way to become a strong self-reliant boy. But such suggestions should be delayed as long as it seems at all advisable, in the hope that the initiative may come from the pupil himself.

The learning in this extra work is done almost entirely by reading and independent work on the part of the pupil. This in itself is important because, after all, our later learning must be done that way. It is true that there are classes and individual tutoring

in contract bridge and that sort of thing, but the practicing engineer, lawyer, doctor, and even the preacher and the teacher, must learn by reading and thinking and experimenting if he is to continue to learn at all.

Individual reports are made on this extra work any time that a pupil has done a substantial amount of it, and the reports are always made on the initiative of the pupils. Reports are never called for, though a "dead-line" for reports on certain topics may be set. However, such work cannot always be done within a definite time limit. In fact, reports on material naturally belonging to the first semester are sometimes received during the second semester and counted as a part of the second semester's work. This is particularly true of a fairly difficult and extensive piece of work in which a capable pupil has become interested. In the majority of cases the reports are made in individual conferences with the teacher, but in some cases a pupil works up a "talk," which is presented to the whole class or to a mathematics club, in case there is one in the school.

Experience has shown that about one-third the members of an ordinary class will undertake some extra work and about one-fifth will do a very considerable amount of it. The amount of extra time required on the part of the teacher is not great. Those who do the work are abler than the median ones and get along with far less help. A brief suggestion, often a single remark, will suffice to clear up a difficulty where a less able pupil would require elaborate and repeated presentation and guidance. In many such cases the best help that can be given is to say: "I really believe you can figure this out for yourself," although with another type of pupil this would result in hopeless discouragement. In fact, herein lies one very considerable element of value of this kind of work. Explanation and help that are adequate for the less able pupils constitute over-explanation for the abler ones.

The effect upon these abler pupils seems to be excellent. They undertake the extra work with greater enthusiasm because it is extra. They acquire experience in dealing with questions requiring persistent and repeated attempts and careful thought and analysis, which they never meet when working on material adapted to the class as a whole. They work much more independently than they otherwise would. They work on large units and accumulate a good deal of material, instead of going to the teacher with each little

assignment. They get some experience in guiding their own activities instead of doing assigned tasks when they are told to do them and in a manner more closely prescribed. Work of this kind extended over a number of years has an astonishing effect upon a student. Of late years my own experience has been largely with college students, and there the effect is certainly very marked. At the end of even one year a college freshman not infrequently will work up a fairly difficult topic, much too difficult for an ordinary class, and present it in good form in a talk taking up a whole hour. In the experimental work in high school teaching, referred to above, some of the pupils "just eat up" this extra work, as the teacher reports, and some of it is so difficult that I am constantly hesitating to put it in. I am perfectly certain that a text including this material in the main course would promptly be placed on the shelf as being too difficult.

We have heard much complaint about the numerous failures in elementary algebra. It is my belief that, more or less unconsciously, in order to provide work that shall be fairly adapted to the abler student we have pitched the level of difficulty too high for the median and less able student. Is it not possible that the solution lies in making the main course simpler than it now usually is, so that the failures will be fewer, and at the same time providing work of a more difficult character which will be better adapted to the testing of the capacities of the abler ones than is the work which now proves too difficult for such large numbers? Individual capacities differ even more greatly, I believe, than is commonly supposed—certainly more greatly than is now provided for in our usual methods of handling this problem.

The plan, once widely advocated and tried in practice, of dividing classes into sections according to ability has not, I believe, met with general favor. To set pupils apart formally and designate them as superior, median, and inferior seems to go a little "against the grain" in our democratic way of thinking. The method here described avoids this unpleasant situation and at the same time provides for the abler pupils the kind of educational experience to which they are entitled.

Conclusion. By simplifying the main course in such ways as are suggested above it is possible to meet the requirements in elementary algebra now made by our general educational situation and at the same time so to lighten the load upon the median pupil

that it will be possible to build into the course as integral and vital parts some of the more general and less formal aspects of the subject. Important among these is the functional aspect of formulas and algebraic expressions in general. I believe the experimental evidence now available warrants the conclusion that the median student can grasp the subject to the extent indicated in this paper and that the wider outlook and more general understanding thus acquired justifies the expenditure of the necessary time and effort.

The abler pupils benefit much more by the proposed plan. They acquire surprisingly clear ideas of the functional relation and of the behavior of such functions as are suggested in this chapter.

By such attention to functionality as has been suggested I believe the course in algebra is made of wider significance than is the case when the work is confined almost wholly to the technique of algebraic operations, necessary as this technique is for those who are to make normal use of elementary algebra. It does, I think, open the pupil's eyes to some extent to functional relations in the world in which he lives and it also opens his eyes to the possibility of representing many of the most important of these relations by fairly simple algebraic expressions and formulas.

THE FUNCTION CONCEPT AND GRAPHICAL METHODS IN STATISTICS AND ECONOMICS

By W. LIETZMANN

University of Göttingen

1. Purpose. This chapter does not aim to give a complete course in statistics and economics, utilizing the function concept and graphical representation. We intend, rather, to show by isolated examples the extent to which, in the curriculum in arithmetic and algebra, we can borrow materials useful in life, and for this reason yielding important information for the school, which are available for a comprehension of the function concept and for graphical representation.[1] How far the teacher of junior high school arithmetic or senior high school algebra can incorporate such examples in his course, I must leave to the teacher himself.

2. The function concept. I must here insert a few remarks from the standpoint of mathematics. In algebra, we run only too easily into the danger of limiting the function concept to such special functions as lend themselves to analytical expression. On the contrary, let us now go on from the general concept which is due to Dirichlet: within any interval, say from a to b, for all or only for certain numbers—the arguments—the functional values are uniquely defined by arbitrary symbols.

3. The mean of the values of a function.[2] The following readings were made from a thermometer in the course of a day:

Time	4	8	12	16*	20*	24*
Temperature . .	20°	38°	51°	56°	45°	30° F.

* [I.e., 4 P.M., 8 P.M., 12 P.M.]

To give a clear picture of this number relation, I can represent the several numbers of the two rows by bars in such a way that their

[1] I refrain from an account of the literature on statistics and on economics. There are countless works which give material for further examples of the discussion which follows. For what pertains to the side of mathematics, I content myself as touching all extensions of this, to refer to W. Lietzmann, *Funktion und graphische Darstellung*, Breslau, Hirt 1925. This discussion I have followed here more than once.

[2] [*Mittelwert.*]

length in millimeters will express the numbers. Instead of these bars, we might also use rectangles, all having the same width but with their lengths depending upon the observed numbers. In Fig. 1, these rectangles are arranged next each other, thus forming a frequency polygon.[3] The upper row of the table furnishes the argument, the lower row yields the associated value of the function.

We now face the problem of finding the mean. This as we know is done by adding all the numbers, thus

$$20 + 38 + 51 + 56 + 45 + 30 = 240,$$

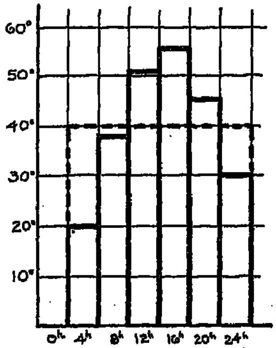


FIGURE 1

and dividing by the total number 6. We find the mean to be 40. In the figure, this value is indicated by a dotted line. The computation presents no difficulty in this example. But finding the mean of many values is exceedingly tedious. It can, however, be simplified. The mean signifies that the part [of the figure] which extends above the dotted line should be equal in area to the part which is lacking in the rectangle cut off by the dotted line. Accordingly, we can estimate the value of the mean provisionally. Naturally, we will not expect to happen on precisely the right value. Suppose we guess 35; then we find the following differences (I make use of the concept of a negative number at this time),

$$-15 + 3 + 16 + 21 + 10 - 5.$$

The sum of these numbers is

$$-15 + 3 + 16 + 21 + 10 - 5 = 30.$$

Thus the correction to be applied to the estimated mean is $30 \div 6 = 5$. Accordingly the mean is $35 + 5 = 40$

4. The mean value of a continuous function. Our procedure in finding the mean temperature for a day based on six readings is most unreliable. The values of the functions assigned to the arguments that have not been given, remain entirely undetermined. If we wish to improve this representation of the mean, we should make

[3] [Literally a Staffeldarstellung, a ladder or step diagram.]

the readings at shorter intervals. The best results are to be obtained if we read the curve from a self-registering thermometer and we make this the basis of our computation. Thus the step diagram in Fig. 2 has now also become a surface whose top is closed by a continuous curve. The method of computation which we used previously, now fails us. Which value shall we choose? Even here, however, the graphic scheme leads us to the goal. The problem is to find the location of the dotted line so that the parts of the surface which overtop this shall be just as large as those which have been added to complete the rectangle [literally, for the creation of the rectangle]. This is a geometric problem. To be sure, a very rough way to solve this problem is as follows: Cut out the surface that is bounded above by the curve and below by the axis and weigh it. Then determine the weight of a rectangle whose width is the width of this piece of surface and whose length is 10 cm. From the relation of the two weights, the position of the dotted line and accordingly the value of the mean are determined. I must leave to the students the practical execution of this, making the procedure trustworthy to a specified degree by the choice of uniform paper that is not too thin. The teacher will then add that mathematicians have invented a practical apparatus (the planimeter) to measure the areas of surfaces, a problem which must often be solved in various sciences (geography, thermodynamics, etc.).

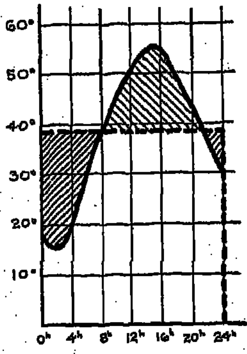


FIGURE 2

The teacher will have no lack of examples in which the determination of the mean is of importance whether it be that he draws upon the different branches of science or that he finds them in daily life, especially life in school. The pedagogical value lies in the fact that in using the shortened computation of the mean, we recognize the value of the introduction of negative numbers. Naturally, the teacher will also select such examples as those in which negative numbers appear among the values of the functions and where, accordingly, the curve drops below the axis. Especially valuable are cases in which, at the first estimate, we cannot tell whether the mean lies above or below the axis. The fact that we make a bold transition from the step diagram to the curve and to the surface concept

will not disturb the younger pupils. Later when the integral calculus is taught we will return to this transitional process which we will encounter again in what follows.

Number of Mistakes	Number of Pupils
0	2
1	5
2	8
3	4
4	1

5. Frequency curves. Suppose that 20 pupils are given an arithmetic assignment of 4 problems. Each wrong answer counts as a mistake. The adjacent table shows the results. We may represent the results in a frequency curve either in a bar diagram (Fig. 3) or in polygon form (Fig. 4). In the polygon form, notice that for reasons which will presently be made clear, the argument should be increased by 1, both to the

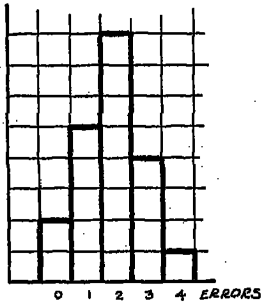


FIGURE 3

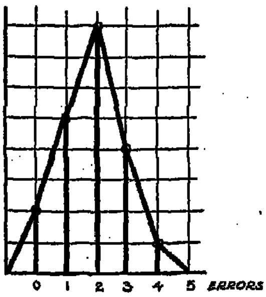


FIGURE 4

left of the 0 and to the right of the 4, the corresponding values of the function being zero. Thus the polygon diagram begins and ends at the axis. We might also transform this into a problem in mechanics [literally, give a mechanical picture of this distribution] by letting a weight which corresponds to a value of the function be attached to the proper value of the argument as is indicated in Fig. 5 by the arrows. In such a case, we actually speak of a weight associated with a particular argument rather than speaking of the value of the function.

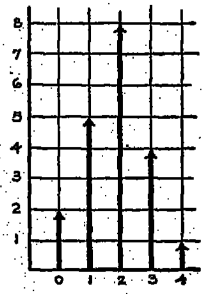


FIGURE 5

Ordinarily, when we deal with the distribution of a multitude of things, we call the number the total of the distribution [*Kollectiv*] after some measurable characteristic which is expressed by means of the argument. In our case, the assignment represents the total

number; the number of mistakes constitutes the value of the argument. To each argument there belongs a frequency. In general, we will denote the values of the argument by $x_1, x_2, \ldots, x_k$, and the associated values of the frequencies by $n_1, n_2, \ldots, n_k$. Hence, the total of the whole comes to N which is equal to $n_1 + n_2 + \ldots + n_k$.

At this point, there is the important concept: In the case of the bar diagram and in the case of the polygon, the area of the surface constitutes a measure of the total, while in the case of the diagram of the weights, the total weight measures the total number, N.

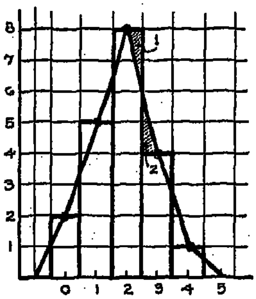


FIGURE 6

In the case of the bar diagram, this is self-evident, as is true also for the diagram of weights. Fig. 6, which shows the transition from the bar diagram to the frequency polygon shows that it is true for the polygon also. [To make the polygon], small right triangles are cut from the bar diagram, but exactly the same number are added. In fact, for each that is cut off a congruent one is added (cf. triangles 1 and 2). The reason for our continuing the polygon to the axis as we did will now be understood.

Examples of frequency curves present themselves in countless number. It is desirable to explain the manifold variety of forms by means of examples in which the curves do not all have one highest point [i.e., some may be multi-modal]; or in which they consist of broken curves where for certain values of the argument in a given interval the frequency is zero; or where instead of the uphill-downhill (*bergauf-bergab*) of our example, the curve exhibits only a rise or only a fall or perhaps an up-and-down character.

6. The mean argument.[4] In the case of the frequency curve, the question of the mean of the values of the function is of no importance but the questions concerning the mean argument are important. In our example, we ask how many errors are made on an average. A person who says 2 because some made 0 or 1, and others, 3 or 4 mistakes has not considered that the errors are not equally distributed and that, therefore, the arguments are to be considered according to their "weight." Accordingly, we form the sum of the

[4] *Argumentdurchschnitt.* [We should say simply the mean.]

products of the arguments and of their corresponding frequencies as follows:

$$0 \cdot 2 + 1 \cdot 5 + 2 \cdot 8 + 3 \cdot 4 + 4 \cdot 1 = 37.$$

Dividing by the total frequency, which is 20 in this case, we obtain the mean of the arguments, that is $\frac{37}{20} = 1.85$. In general, the mean of the arguments is

$$A = \frac{n_1 x_1 + n_2 x_2 + \ldots + n_k x_k}{n_1 + n_2 + \ldots + n_k}.$$

Hence the theorem: In the case of the representation by weights, the mean argument is the argument belonging to the center of gravity of the figure as is evident by the expression

$$An_1 + An_2 + \ldots + An_k = n_1 x_1 + n_2 x_2 + \ldots + n_k x_k.$$

Hence

$$n_1(A - x_1) + n_2(A - x_2) + \ldots + n_k(A - x_k) = 0.$$

Since we can represent the $n_1, \ldots n_k$ as forces and since the $(A - x_1) \ldots (A - x_k)$ with their proper signs represent the distances from A to $x_1, \ldots, x_k$, we may say that, the signs being taken into account, the sum of the products of the forces by the distances of the forces is zero.

The mean argument is an important characteristic of a frequency distribution even when this is expressed as an equation. If the 20 pupils write another assignment of four exercises, a comparison of the two means answers the question as to which is done the better. The comparison is equally good, however, if instead of 20 pupils a greater or a less number take part in the work, or even if the same piece of work is given to other pupils. For this reason, the mean argument is of the greatest importance in all evaluations of tests.

In school, countless things may be represented by frequency curves. The mean argument should be determined and its significance should be discussed. When a teacher grades any exercise with the five marks A, B, C, D, E, if a normal distribution is assumed, he should take care that the grades are so arranged that C is the mean argument. There are teachers whose judgments are in no way "symmetric," whose grades tend to have their mean as B or as D. On this basis, we can distinguish the three types of

teachers almost immediately, and the students are likely to be aware of this also.

7. **Summation curve.**[5] For many problems, the so-called summation curve is better than the distribution curve. From the table on page 77, we can read the following table:

Number making 0 mistakes	2 pupils
Number making 1 mistake or fewer	7 pupils
Number making 2 mistakes or fewer	15 pupils
Number making 3 mistakes or fewer	19 pupils
Number making 4 mistakes or fewer	20 pupils

We may represent this by bars or by the polygon method (Fig. 7 and Fig. 8) and we observe that the last bar or the last ordinate of the polygon shall here represent the 20 records. The curve is steadily rising. There exist certain correspondences of shape between the

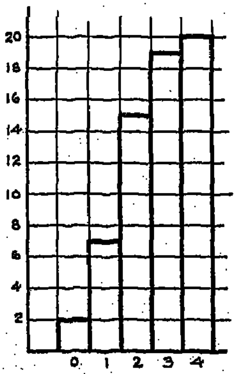


FIGURE 7

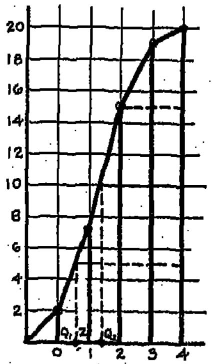


FIGURE 8

distribution curve and the summation curve and an opportunity should be given to the students to examine the various types. Thus to the up-and-down course of the frequency curve, there belongs a summation curve which is at first slow, then quicker, then slow again, somewhat in the form of an integral sign set on a diagonal.

8. **The median value.** We are seeking the argument which belongs to the value of the function that exactly halves the total group. This argument is called the median. In the case under consideration, taking the total number of cases as 20, 10 is situated somewhere in the 7th to 15th cases lying in the interval of two mistakes [or

[5] [I.e., the curve showing the number not exceeding a given score.]

fewer]. If we use linear interpolation, the difference 8 corresponds to 1, the difference 3 to x, and x is $\frac{3}{8}$. As the median, therefore, we have $1\frac{3}{8}$. Graphically, we find this from the polygonal representation of the summation curve. Halve the last ordinate. Through this point, draw a line parallel to the axis. Read the value of the argument of the point of intersection of the parallel line with the polygon.

The median is not so important as the mean. It is, however, often used in a more extended form in the following way. As was done above, the total distribution is brought into a rank order and is then halved. We can also divide the distribution into four parts. We say that the total is quartered. Thus we easily obtain the three quarter points Q_1, Z, Q_2. Then half of the cases lie between Q_1 and Q_2, and, indeed, half of the cases lie on either side of Z. In our example, $Q_1 = \frac{3}{8}$, $Q_2 = 2$.

As a further example from the study of tests: A test is so standardized that 25% (in the sense of the test) are above normal, 50% are normal, and 25% are below normal. We find the proper points, that is the arguments, by finding the quartiles.

9. The binomial normal curve. We take our point of departure from a problem which seems somewhat strange to us. The point A is the corner of a rectangular system of streets indicated in Fig. 9. A pedestrian starts in the direction of the arrows and never turns back. The question is, how many ways are open to him at the intersections of the streets? We see that the intersections of the streets form horizontal rows. In the row next the beginning there are two intersections. In the next row there are three, in the next four, in the next five, if we count those intersections which lie at the edge of the area and which are formed by three instead of the usual four streets. To the points of the first row, there leads but one path. To the points on the edges of the second, there is but one approach. To the central point, however, there are two approaches.

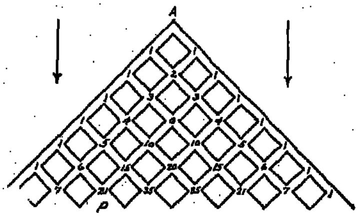


FIGURE 9

In the third row, the situation is again a different one. Here as before, the edge points have but one approach as is the case in each of the rows. In order to determine the number of ways leading to the

other series of points, we examine both approaches. To the first point, there is one approach. To the second, two; and to the third, three. We can proceed in this manner. In the figure, the number of ways is represented up to the seventh row. We can easily supply the number for each point of the next row. We need merely to take note of the two approaches that lead to it. We may obtain the new figures by means of addition. I should add that the entire arrangement of the center of the streets is symmetric.

This is an application of the Pascal triangle as a consequence of the binomial coefficients.

We can now put the problem in another form. A man goes aimlessly through the maze of the streets in the direction of the arrows. What is the probability that he will reach a certain point P in, let us say, the seventh row? In mathematics, the term "probability" has a very definite significance, namely, the fraction whose numerator is the total number of favorable cases and whose denom-

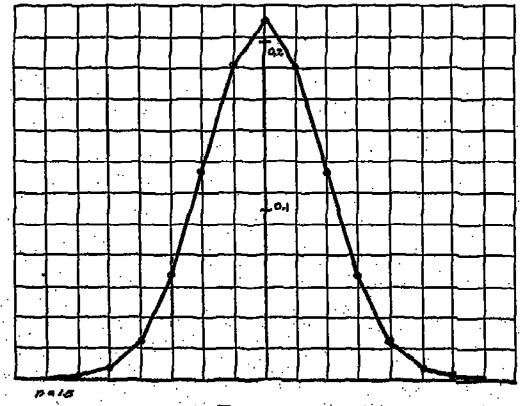


FIGURE 10

inator is the total number of possible cases. The number of favorable cases is already on the chart; for the point P it is 21. The number of all possible cases may be obtained by adding all the numbers of the seventh row. This is 128. It is no accident that this is exactly a power of 2. From every point of the sixth row, two ways branch off. The points of the seventh row have exactly twice as many possible approaches as have all the points of the sixth taken together. The points of the sixth again have double those of the fifth row, those of the fifth have double those of the fourth,

and so on. Accordingly, the total in the fifth row is 2^5; in the sixth, 2^6; in the seventh, 2^7 or 128.

We will now represent the possibilities for any row, say the fifteenth, graphically. This appears in Fig. 10. We shall now test our

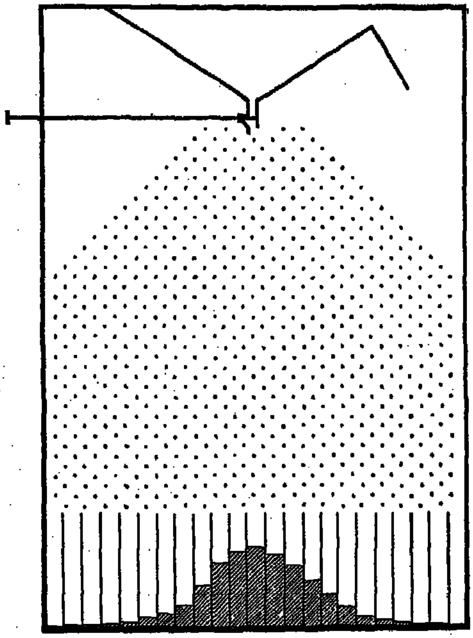


FIGURE 11

theoretical discussion by a practical experiment. To be sure, it would be very tedious to have a man carry out the scheme a large number of times. Instead of this we can carry out the scheme with many people simultaneously, or at very short intervals. Instead of using people, we shall use spherical shot. For the blocks of houses, we shall substitute nails driven into a board. The board stands upright or else is slightly inclined, so that large numbers of shot from a container meander down over it between the nails and are caught in a suitable manner. Fig. 11 shows such a board, which was first constructed by Galton. The thing that is surprising for the mathematician as well as for the layman is that probability curves are formed, although the way each shot takes is governed wholly

other series of points, we examine both approaches. To the first point, there is one approach. To the second, two; and to the third, three. We can proceed in this manner. In the figure, the number of ways is represented up to the seventh row. We can easily supply the number for each point of the next row. We need merely to take note of the two approaches that lead to it. We may obtain the new figures by means of addition. I should add that the entire arrangement of the center of the streets is symmetric.

This is an application of the Pascal triangle as a consequence of the binomial coefficients.

We can now put the problem in another form. A man goes aimlessly through the maze of the streets in the direction of the arrows. What is the probability that he will reach a certain point P in, let us say, the seventh row? In mathematics, the term "probability" has a very definite significance, namely, the fraction whose numerator is the total number of favorable cases and whose denom-

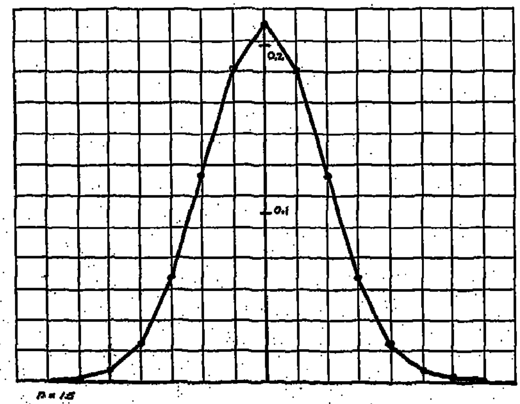


FIGURE 10

inator is the total number of possible cases. The number of favorable cases is already on the chart; for the point P it is 21. The number of all possible cases may be obtained by adding all the numbers of the seventh row. This is 128. It is no accident that this is exactly a power of 2. From every point of the sixth row, two ways branch off. The points of the seventh row have exactly twice as many possible approaches as have all the points of the sixth taken together. The points of the sixth again have double those of the fifth row, those of the fifth have double those of the fourth,

and so on. Accordingly, the total in the fifth row is 2^5; in the sixth, 2^6; in the seventh, 2^7 or 128.

We will now represent the possibilities for any row, say the fifteenth, graphically. This appears in Fig. 10. We shall now test our

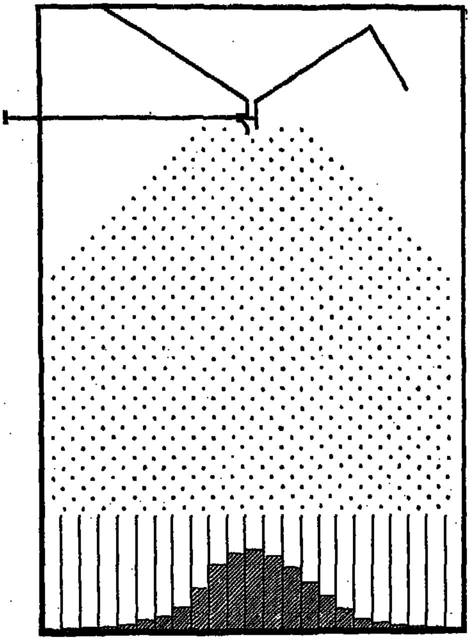


FIGURE 11

theoretical discussion by a practical experiment. To be sure, it would be very tedious to have a man carry out the scheme a large number of times. Instead of this we can carry out the scheme with many people simultaneously, or at very short intervals. Instead of using people, we shall use spherical shot. For the blocks of houses, we shall substitute nails driven into a board. The board stands upright or else is slightly inclined, so that large numbers of shot from a container meander down over it between the nails and are caught in a suitable manner. Fig. 11 shows such a board, which was first constructed by Galton. The thing that is surprising for the mathematician as well as for the layman is that probability curves are formed, although the way each shot takes is governed wholly

by chance. The degree of approximation, however, depends on the quality of the board and the size of the shot.

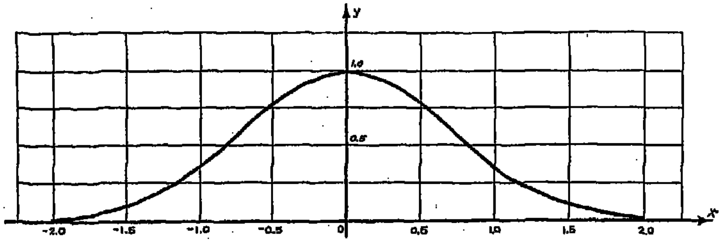


FIGURE 12

10. The Gaussian curve. By means of a transition which is not possible without a considerable knowledge of the infinitesimal calculus, we obtain a continuous curve, the Gaussian curve of error. For a simple example we have the equation $y = e^{-x^2}$. Fig. 12 shows the course of the curve which is clearly recognized as the limiting case of the binomial distribution curve. The noteworthy thing now is that in the case of certain statistical studies, curves of the type appear exceedingly often, and if bar diagrams alone are present, these are themselves close approximations to the Gaussian curve. Fig. 13 shows the distribution of the I. Q.'s of 905 children from 5 to 14 years, selected at random, the table being according to Terman. The resemblance to the normal curve appears to be very close.

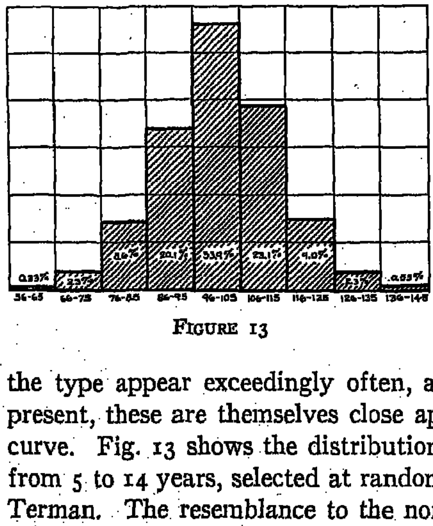


FIGURE 13

11. Other distributions. I will mention at least one statistical example which does not resemble the Gaussian curve, because of its great practical significance. Life insurance has for its basis the assumption of a statistical tabulation of the death ratio. From countless cases, mortality tables have been established. The actuary has discovered statistically how many of 100,000 twenty-year-old men will live to be 21, 22, . . . to 90 years. This material, still somewhat dependent upon the special circumstances of the persons under consideration, which we shall not discuss here, is put together

in a mortality table. Fig. 14 shows the graph of such a table. It is somewhat in a diagonal line. We shall see that in general, the curve diminishes first slowly, then ever faster; though after 70 years, the losses become slower again. If we represent the deaths in a single year by a graph, we obtain Fig. 15. It would appear that after the age of 70, the probability of dying always becomes smaller. This, however, is an error as the absolute number of yearly deaths among the first 100,000 twenty-year-olds naturally becomes smaller in the seventieth or eightieth years, for then only comparatively few are still living. If we compute the percentage of those reaching each age who will die within a year, we obtain an entirely different curve, the curve of the probability of death. Fig. 16 shows this. From this, it appears that until about the age of 50, the probability of death rises slowly; from that time on it rises more and more sharply to the peak.

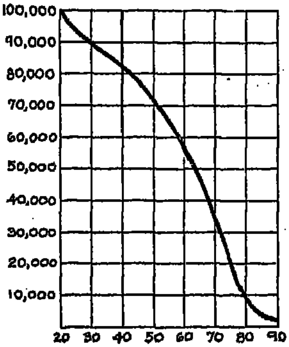


FIGURE 14

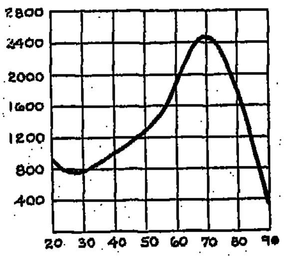


FIGURE 15

Although our curves have lost the appearance of polygons, we must actually picture all these statistical curves which are based on counting as having the characteristics of polygons, with the end points of their sides lying in very close proximity.

To the instructor must be left the selection of the statistical material to be used as examples—material from school life, from the life of the community, from the life of the state. Another suggestion: a wise rule concerning the proper age when a man and woman should marry. Add 10 years to the age of the man and take half of the sum. In this manner one may obtain the proper age for the wife. If x is the age of the man and if y is the age of the

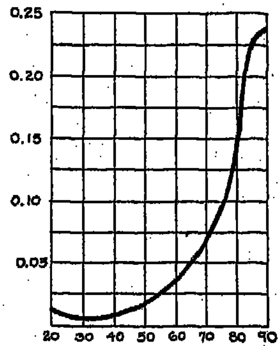


FIGURE 16

woman, then $\frac{x+10}{2}=y$; $y=\frac{x}{2}+5$. This is shown graphically in Fig. 17. Aside from the fact that this function has no significance for small values of x, and aside from the fact that there are no negative values, the very interesting question arises as to how

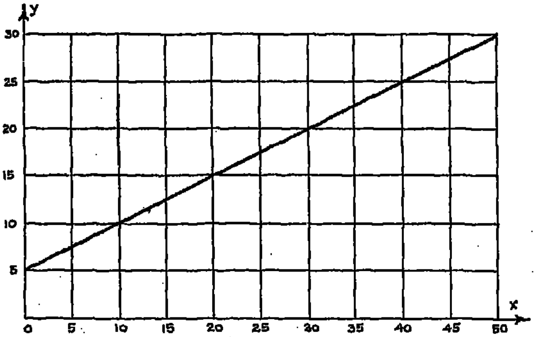


FIGURE 17

far this rule proves itself to be even approximately true compared with the statistics of a city or of a rural community.

I here terminate my treatment of statistical concepts in the mathematics curriculum. He who wishes to proceed further will be able to include the concept of taxation. In the study of advanced algebra and the differential calculus, we can also treat the Gaussian curve by the method of least squares and show where its two points of inflection lie—points which again are connected with the problem of taxation.

12. Graduated costs. The characteristics of a graduated cost are that certain charges fixed by law are constant when the argument lies within certain intervals, but jump suddenly at the limits

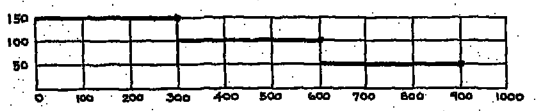


FIGURE 18

of the interval. To this group belong postage costs, countless duties and tariffs, zone charges on railroads, and many others. Let us consider a simple example: As a consequence of favorable economic conditions, workmen who have been receiving a low salary get an increase. An income below \$300 is increased by \$150; an income

of from \$300 to \$600 receives \$100; and those from \$600 to \$900 get \$50. Above this point, there is no increase. The graphic representation in Fig. 18 shows why the expression graduated[6] costs is a happy choice. In the case of such a situation, we must decide which value is to be taken at the discontinuities. There are two possibilities, an upper and a lower. In our problem, the upper value is always to be assumed.

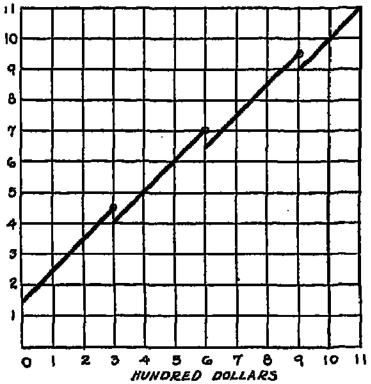


FIGURE 19

Naturally in the case of these problems, the gradations need not necessarily be equally long nor equally high. For example, in the case of taxes, the most peculiar arbitrary conditions appear in the gradation of the steps. They cannot be represented otherwise. Our graduated scale seems very sensible, but we shall soon see that it has a very serious fault. Let us investigate the increase the workman has had. As the argument, let us take his former income and as the value of the function, let us take his new income. We shall then have the graph represented in Fig. 19. Now let us turn our attention to the discontinuities, or, to borrow a figure of speech from geology, the *faults*. The workman with an original salary of \$300 now receives \$450 since his increase amounts to \$150. The man with \$301 receives an increase of only \$100. His new income is therefore \$401. Because he received \$1 more formerly, he now receives \$49 less. This is naturally a great injustice and it is repeated in the case of each discontinuity. A way of avoiding this injustice is shown in Fig. 20

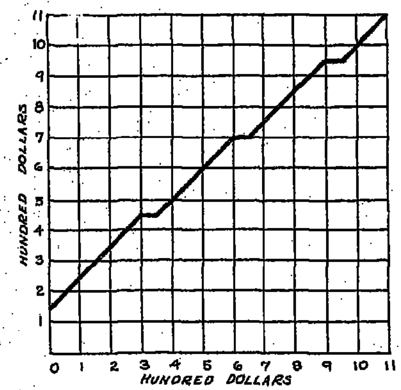


FIGURE 20

[6] *Cf. gradus*, a step.

in which the breaks are replaced by horizontal line segments. In expressing this solution in words, the legislator does not have so easy a task as we have with our graphical representation. He will say something like this: For incomes up to $300, there will be an increase of $150. In the case of larger incomes including $600, the increase will be only $100, except that it must be at least enough to make the final amount not less than what one would receive on an income of $300.

Even the new solution is in no way ideal. Bridging the gaps through these gradations brings about a certain continuity for the polygon, but it nevertheless contains a big injustice in itself. As the income increases originally, so should the increased incomes increase. We might therefore wish for another solution.

13. Rates per cent. Tax rates do not establish constant costs as due fixed charges but are percentages of income. Again, we will frame a very simple example. Suppose that incomes to $3,000, inclusive, are tax free. Suppose that to $6,000, there is a tax of 10%; to $9,000, 20%, beyond that 30%. What happens above that point does not concern us. Fig. 21 shows this graphically. This is a fixed rate of the type with which we have just become acquainted. What is the state of the incomes remaining after the tax has been deducted? The graph (Fig. 22) can be interpreted at a glance. Here again we have discontinuities. The height of the breaks is not equally great. The line segments have different slopes.

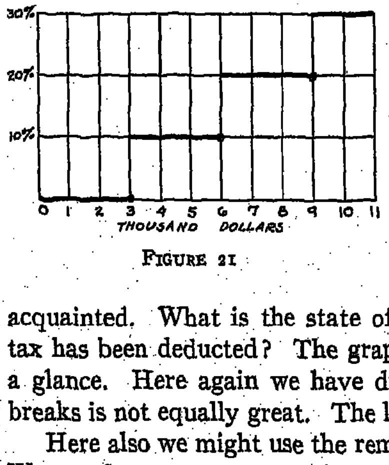


FIGURE 21

Here also we might use the remedy of bridging the line segments. We see that no taxes are paid on $3,000. For incomes greater than that to $6,000, inclusive, 10% is paid. For those above this point to $9,000, inclusive, 20%, etc. Now, the graph of Fig. 23 shows no discontinuities. A continuous, constantly rising polygon has been formed.

The legislator who faces the disputed points can eventually make a substitute for the polygon by a suitably constructed curve which is constantly changing its slope. The tax collector can read the amounts from the graphic representation; for whenever the one figure (the income) is given, the tax is the difference between the

dotted line bisecting the angle between the axes, and the curve. Thus we have obtained a continuous curve.

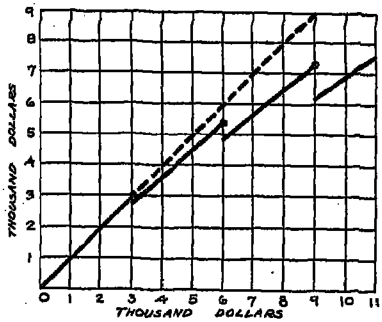


FIGURE 22

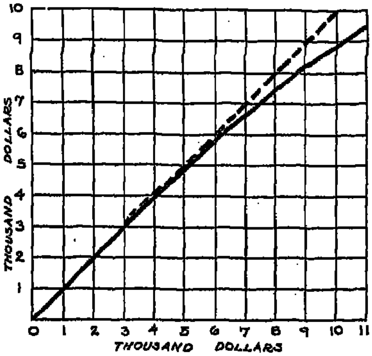


FIGURE 23

14. Discount problems. My last examples bring us again to mathematical curves. Smith gives Young a note for K dollars to be paid back after n years. Let n be a proper fraction so that we may use simple interest at say $p\%$. Then it is customary for Smith to deduct the interest which is $\frac{Kpn}{100}$ dollars. Young receives only $k = K\left(1 - \frac{pn}{100}\right)$. We readily see that this is not free from criticism. Smith has not actually given Young K dollars but only k dollars. Therefore, he should not have reckoned the interest on K dollars. The following computation is more correct: If Smith gives Young k dollars to-day, in n years this will come to $k\left(1 + \frac{np}{100}\right)$ dollars. This is the amount K of the note. Accordingly, $k = \frac{K}{1 + \frac{np}{100}}$.

The situation becomes clearer if we set the problem down graphically. As an example, let us take $p = 10$. Since K is a constant factor in both cases, let us put it down as 1. Then k becomes a function of the time which we will now call t. Then at one time $k = 1 - \frac{t}{10}$ [i.e., by bank discount], at another $k = \frac{1}{1 + \frac{t}{10}} = \frac{10}{10 + t}$ [i.e., by true discount]. The graph of the first function is a line and that of the second is a hyperbola, the line being tangent to

the hyperbola at the point where the hyperbola cuts the y-axis. Fig. 24 shows the facts of the case. The graph is, of course, limited to the first quadrant. In the course of the first years, the hyperbola and the line fall comparatively close together, the difference between them becoming greater and greater. After ten years, the value of the first function is zero. Beyond that it is negative, while the second function always remains positive. Even though the injustice of the first function is noticeable only for large values of t, nevertheless, it is actually present from the beginning.

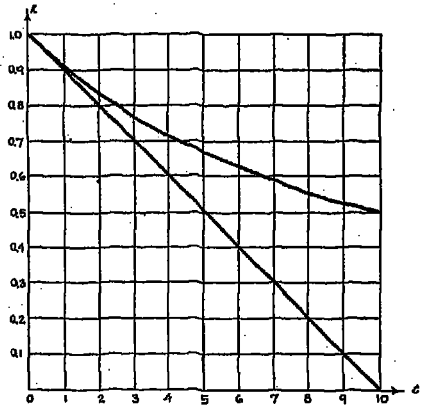


FIGURE 24

15. Exchange. Every stock market notes the course of exchange on money. We note that last autumn according to the information in the Berlin papers the Swiss and French francs were noted as follows:

$$1 \text{ Swiss franc} = 0.8169 \text{ RM.}$$
$$1 \text{ French franc} = 0.1646 \text{ RM.}$$

On the same day Zürich noted

$$1 \text{ RM} = 1.2245 \text{ Swiss francs}$$

and Paris,

$$1 \text{ RM} = 6.0625 \text{ French francs.}$$

Now $0.8169 \cdot 1.2245 = 1.0003$

and $0.1646 \cdot 6.0625 = 0.9979$

Both times, the quantity on the right is approximately equal to 1. Is this by chance? Let us denote one kind of money by A, another kind by B. Then one stock market notes:

(1) $1A = xB$

Another says (2) $1B = yA$

From (1) it follows that $yA = xyB$

From (2) $yA = 1B$

Accordingly, $xy = 1$

This functional equation, therefore, connects the two rate curves with each other. The graphic picture of this relationship is an equilateral hyperbola (Fig. 25). The student will be able to examine the different quotations of the dollar in the different stock exchanges and the corresponding value of the foreign money in New York to see how far this law of exchange fits the case day by day.

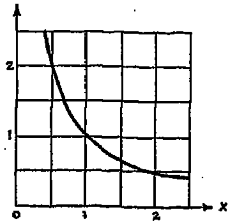


FIGURE 25

16. *Compound interest.* When one lends capital for a long period, it increases at compound interest. If the interest is added yearly as in a savings bank, then the graph of the amounts at the close of each year consists of points on a steadily rising curve as is shown in Fig. 26, which represents an initial capital of \$100 with interest at 5%, the scale on the y-axis beginning at 100. The points lie on an exponential curve, for if t, the variable time in years is the argument, then the total amount at the end is, as is well known, $k = 100 \cdot 1.05^t$. If some of the capital is withdrawn within one year, the bank usually begins to pay simple interest for that time from the beginning of the year. If we wish to express this graphically, the points would have to be joined by lines.

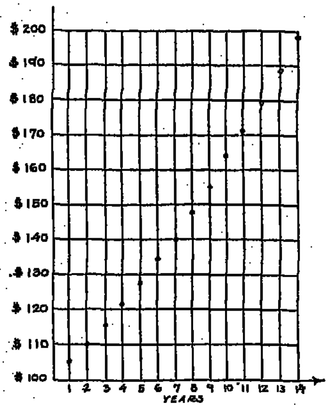


FIGURE 26

But this does not actually represent the facts of the case. If I withdraw some money at 8 A.M. one day, I receive exactly as much as if I withdraw it at 4 P.M., yet the time has increased continuously and I should have received more interest. The fine structure of our connecting lines reveals itself as a curve of 365 steps and in leap year 366. Further, if one considers that the bank is closed for certain hours on certain days and that the capital cannot then be obtained at all and that perhaps the months are counted as having thirty days and that perhaps only the even dollar draws interest, and other things of the same sort, then the fine structure of the curve becomes very complicated. There is even an influence on the course of the total as in the last assumption, although this is slight in the case of large amounts.

17. Conclusion. An objection, which we hear frequently, to the inclusion of practical applications in the teaching of arithmetic and algebra is that the explanations necessary for an understanding of the applications take too much time—time which is lost to the actual purpose of the instruction, namely mathematics. I believe our examples demonstrate that this danger is not so great. For if one chooses suitable examples, the content lies so very much within fields that the pupil knows from his practical experience of life or from what other subjects have taught him that the little that is still to be added can be given without loss of time. More than that, however, even when explanations must be given to problems from statistics and from community life, these are quite suitable, for I believe that even mathematics should contribute from its materials of instruction to the education of the pupil as a citizen; education is the most effective when it can be founded on knowledge, especially on a knowledge that rests on numbers.

MEASURING THE DEVELOPMENT OF FUNCTIONAL THINKING IN ALGEBRA

By E. R. BRESLICH

School of Education, University of Chicago, Chicago, Illinois

I. Characteristic Properties of the Function Concept

Historical summary. Algebraic functions have always been an important phase of mathematics. In the early stages of the secondary school the work with functions in secondary mathematics was limited to discussion, manipulation, substitution, and evaluation of powers and polynomials involving one or several variables. Later, graphical work enriched the subject to the extent that solutions of equations were introduced and supplemented by a study of functions.

In 1893 Professor Felix Klein of Germany in an address before the International Congress of Mathematicians at its meeting in Chicago directed forcefully the attention of the teachers of secondary school mathematics to the possibility and need of developing functional thinking in their courses.

The subject was further emphasized in 1906 by Professor E. H. Moore [1] of Chicago who expressed the view that the function concept should come to play a fundamental rôle in the reorganization of elementary mathematics.

In December, 1921, Professor E. R. Hedrick [2] addressed the Mathematical Association of America at its meeting in Toronto on the subject of functional thinking. He worked out more fully the ideas of functional relations expressed by the National Committee on Mathematical Requirements.[3]

In 1928 an article on the subject of functional thinking appeared in the *Third Yearbook of the National Council of Teachers of*

[1] Moore, E. H., "The Cross-Section Paper as a Mathematical Instrument." *School Review*, XIV (May, 1906), 317-38.

[2] Hedrick, E. R., "Functionality in Mathematical Instruction in Schools and Colleges." *Mathematics Teacher*, XV (April, 1922), 191-227.

[3] *Report of the National Committee on Mathematical Requirements*, 1923.

Mathematics.[4] It gave the results of an analysis of algebra and geometry textbooks, and disclosed that at that time textbooks were not yet making sufficient use of the numerous opportunities for training in functional thinking in algebra and that in geometry even less emphasis was being given to functional thinking. Ways of teaching functional relationships were explained for such topics as verbal problems, representations of quantitative facts in tables, formulas and graphs, equations, proportion, polynomials, variation, geometric figures, and geometric principles. Some evidence was presented which tends to show that pupils of high school level are able to acquire the ability to think functionally.

Within recent years interest in the subject has not diminished. Several discussions relating to the function concept in mathematics have appeared in mathematical journals (a bibliography is given at the end of this chapter). The subject is also being treated more adequately in some of the newer textbooks and in books on the teaching of mathematics. In the present study the writer has undertaken the task of summarizing and classifying the suggestions that have been made or implied in these articles and books, the purpose being to arrive at an analysis of functional thinking as conceived by present-day writers, to list what the writers seem to consider the objectives of functional thinking, and to measure the extent to which the objectives are actually being acquired by high school pupils.

Meaning of functional thinking. Experience with quantitative relationships is not new to pupils when they enter the secondary school. They are familiar with an abundance of situations in which variable quantities depend on others for their values. They know, for example, that the cost of sending a parcel depends on the weight; that the price of a package of pencils depends on the number of pencils; and that the amount of gasoline used on an automobile trip depends on the length of the trip. Recognition of the dependence of one variable quantity on another related variable is considered by writers to be one of the important aspects of functional thinking. Other aspects are: recognizing the character of the relationships between variables; determining the nature of the relationships; expressing relationships in algebraic symbols; and recognizing how a change in one of the related variables affects the values of the others.

[4] The *Third Yearbook* (1928), pp. 42-56.

Thus, the original meaning of the function concept has been broadened. Functional thinking as now taught implies the idea of variable quantity. It is concerned with the relationships which exist between variables, and with the fact that to a value of one corresponds a definite value of the other.

II. How to Attain the Objective of Functional Thinking

Objective of functional thinking. The development of the various characteristics of functional thinking may be set up as the objective to be attained. This will make the teaching of the function concept intelligent and purposeful. Furthermore, it will be possible to construct tests that will yield evidence regarding the extent to which the aims are actually being accomplished by the pupils. The objectives, tests, and results are discussed in this chapter.

Method of developing functional thinking. Dependence, correspondence, relationship, and changes of variables are usually presented in verbal statements, tables, graphs, and formulas. They may be appreciated, studied, and understood long before a formal definition of the function concept is given. Some types are so simple that they may be profitably discussed in the beginning of the secondary school period. Every course, in fact every unit, should be planned to make definite contributions to the development of functional thinking. During the early stages a topical treatment is neither necessary nor desirable. What is needed is not an intensive study of the function concept as a separate topic, carried on to the exclusion of other topics, but wide and frequent experiences distributed over the various courses.

Ability to recognize and understand dependence. Dependence is probably the most familiar aspect of the function concept. The pupil's everyday experiences have taught him early the idea of dependence. He knows that the height of a boy depends on his age; that his earnings depend on the amount of work he does; that interest depends on the length of time his money is left in the bank; and that the distance he travels depends on time and rate of travel. In his school work he should learn that the character of dependence is frequently specified by means of equations and formulas. He should be taught to recognize dependence before the formula or equation is actually discovered. Thus, even before he has learned that $s = (n - 2)\ 180°$, before he has determined the exact char-

acter of the dependence, he should be able to see that the sum of the angles of a polygon depends on the number of sides. He should know that the area of a triangle may depend on the sides, the base and altitude, or two sides and the included angle. Before the quadratic formula is developed, he should realize that values and nature of the roots of a quadratic equation depend on the values of the coefficients.

Mathematical applications taken from other subjects should be made to contribute to the understanding of dependence. Thus, the distance a body falls depends on the time, and the time of vibration of a pendulum depends on its length.

Ability to recognize relationships. The power to sense and recognize relationships is of value in mathematics, in other school subjects, and in everyday life. The pupil should acquire the habit of thinking about relationships. They may be discussed profitably even when it is not possible to express them as precise mathematical laws. Thus, a person's health is related to good food, fresh air, housing conditions, and exercise. A pupil's weight is related to his age. The sales of overcoats, rubbers, and umbrellas are related to the weather; the wheat crops to the rainfall; the cost of a trip to hotel rates; a person's income to his skill, health, and available work; the premium of an insurance policy to the age of the applicant; and the enrollment in schools to the size of population.

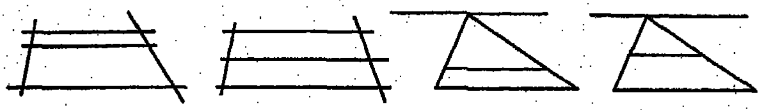
Throughout the mathematical courses the pupil should encounter experiences with relationships. Numerical facts are often organized in tabular form. Familiar examples are the reports of daily temperatures, populations, farm products, and tax distributions. At first the data in such tables may seem to have no connection with one another. However, when they are more carefully examined and interpreted, relationships will be recognized. For example, in a table giving facts about spending the income of a family the various amounts for food, clothing, rent, savings, charity, and amusement may not seem to be at all related, but a little reflection will show how each item is related to the total income as well as to all other items. If an understanding of the relations is to be gained, the facts given in the table may have to be rearranged in order of magnitude, or any two may be compared with each other, with the total, and with the average. The pupil who studies numerical tables in this manner will acquire a better understanding of tabular representa-

tion than one who conceives in the table merely a number of separate facts. He will receive valuable training in searching for and in recognizing relationships.

In algebra the pupil should be taught to recognize relationships involved in problems and exercises because they are needed in solving the problems. Typical examples are the relations between interest, time, and rate; the number of articles purchased and the price paid; the value of taxable property and the taxes; the value of a polynomial and the value of the variable; the cost of a railroad ticket and the distance traveled; and the postage for sending a parcel and the distance and weight.

Through practice in solving equations the pupil should receive training in recognizing relationships. He should acquire the habit of analyzing the processes and operations which are to be employed in order that the value of the unknown may be determined. Thus, in $x + 8 = 12$ he should recognize without difficulty that it requires a subtraction to undo the addition indicated in $x + 8 = 12$, and that division is required to undo the multiplication in $8x = 12$. When called upon to solve equations like $x^2 = 16$, $\frac{1}{2}gt^2 = 10$ and $\sqrt{x} = 4$ he should have no difficulty in choosing the required inverse processes of finding the square root in the first two equations and of squaring in the third. Emphasis on thinking of this type will minimize the habit of thoughtless manipulation of symbols in algebra and will lead to mastery.

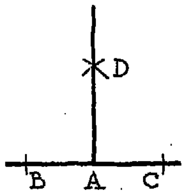
The ability to recognize relationships is no less valuable in geometry than in algebra. The relationships between area of circle and radius, circumference and diameter, the three angles of a triangle, the angle and the intercepted arcs, the segments of intersecting chords and secants, and many other relationships should be clearly understood by all pupils. There is no scarcity of material in geometry for training in functional thinking.

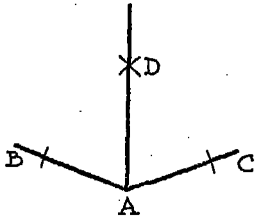


Moreover, the study may be greatly simplified by pointing out relationships between theorems. For example, the pupil should see that the preceding diagrams are related to each other in the sense that they may be regarded as special cases representing the general

principle that if three or more parallel lines are cut by a transversal the corresponding segments of the transversals are proportional.

Equally valuable is the ability to recognize relationships among constructions. Thus, the construction shown in the two figures below of a perpendicular to a line at a given point is identical to the construction of bisecting an angle. The first may be considered a special case of the second.





Ability to express relationships in algebraic symbols. Training in stating relationships in symbolic form is derived from problem solving. Through repeated use many relationships are so thoroughly learned that pupils are able to state them automatically. Illustrations are problems relating to distance, time, and rate; percentage, rate, and principal; centigrade and Fahrenheit; direct variation; inverse variation; mensuration in geometry; and numerous others.

In problems relationships are often implied in single words or phrases and are not always easily recognized. They require much practice. Examples are relationships involved in such terms as: "exceeds," "is greater than," "is less than," "equally distant apart," "several hours late," and "times as great."

Sometimes relationships are illustrated in tables. Thus, the facts of the table

If $x =$	1	2	3	4	5	— — —
Then $y =$	5	10	15	20	25	— — —

are represented by the equation $y = 5x$; the table

If $x =$	1	2	3	4	— —
Then $y =$	48	24	16	12	— —

is represented by $xy = 48$; and the table

If $x =$	1	2	3	4	— — —
Then $y =$	5	7	9	11	— — —

by $y = 2x + 3$. It seems that in courses in algebra the pupils receive much practice in making tables when the equations are given, but that the inverse problem of making the formula when the table is given has received less attention. Typical illustrations of formulas and equations derived from tables are: the sum of the angles of a polygon, the rth term of arithmetical and geometric progressions, the sum of n terms of a progression, and the binomial theorem.

Ability to recognize how a change in one variable affects related variables. The understanding of the way a change in one variable affects others related to it is an important characteristic of functional thinking. Thus, in a linear equation as $c = 2\pi r$, an increase or a decrease in r causes a corresponding increase or decrease in c. If r is doubled or trebled, c is also doubled or trebled. If r is multiplied or divided by 2, c is multiplied or divided by 2. If r is increased or diminished by 4, c is increased or diminished by $2\pi \times 4$.

In a quadratic equation, as $a = \pi r^2$, or in a cubic function, as $v = \frac{4}{3}\pi r^3$, a change in r causes a very different change in a or v. The pupil should readily answer such questions as:

If in $a = \pi r^2$ and $v = \frac{4}{3}\pi r^3$ r is doubled, divided by 2, increased by 4, diminished by 3, what are the corresponding changes in a and in v?

In $i = 0.05\ pt$, how is i changed if p is doubled and t remains constant? How is i changed if p and t are both doubled? How is i changed if p is doubled and t is trebled?

In $z = \frac{x}{y}$, how does z change if y remains the same and x is increased, decreased, multiplied by 2, or divided by 3? If x remains the same and y is increased or decreased?

If the diameter of a cylindrical jar is doubled and the length remains the same, how is the content changed?

Will two one-inch pipes be able to carry the flow of water coming from one two-inch pipe? Give reason.

If one side of an equation is multiplied by 3, divided by 5, increased

by 1, diminished by 9, what corresponding changes should be made in the other side?

The study of the changes in geometric figures offers training in functional thinking. For example, if the length of the upper base of a trapezoid is decreased until it becomes zero, the trapezoid is changed into a triangle. At the same time $A = \frac{1}{2}h(a + b)$, the formula for the area of a trapezoid, is changed to the formula $A = \frac{1}{2}hb$ for finding the area of a triangle, since a decreases until it becomes zero. Similarly, if a, b, and h are all changed until they become equal, the formula changes to $A = a^2$, and the trapezoid is changed into a square. If a becomes equal to b but not equal to h, the formula reduces to $A = bh$ and the trapezoid changes into a parallelogram. The study of the changes of geometric figures helps the pupil discover important algebraic relationships which the study of the separate theorems may fail to disclose.

When numerical facts are presented in tabular form, the pupil should study the table as a whole and note how the numbers in the table change. This they do not always do. Often the only use made of the table is to determine corresponding values; for example, to obtain points for making a graph, to find the particular weights corresponding to given ages, or to find the values of trigonometric ratios corresponding to certain angles. Pupils may use the trigonometric table day after day without noting the important fact, which is readily discovered by looking at the table itself, that the sine function changes from zero to one as the angle changes from 0° to 90°. Such matters are missed by the pupil who has not acquired the habit of paying attention to the changes in the tables that he uses in his work.

In interpreting graphs it is important to note the changes that are pictured. Thus, in finding the meaning of an hourly temperature chart the pupil should ask: When was it warmest? When was it coldest? During which hour was the rise or drop least? When was the change greatest? In the graph of $y = mx$, he should note the changes in the steepness of the line as he changes the value of m. In following the changes of the ordinate in the graph of $y = ax^2 + bx + c$, he should find the zero values of the function and the maximum or minimum value. In the graph of $y = \sin x$, he should see readily the changes of the sine function as x changes from 0° to 360°. The graph helps him retain a mental picture of

the facts that sin x changes from 0 to 1 to 0 to -1 to 0, and that it is positive in the first two quadrants and negative in the third and fourth. The study of the changes in the graph will add much to his understanding of the nature of the sine function.

III. Making the Function Concept the Unifying Factor of Mathematics

Using the function concept in organizing mathematics. Some writers consider the function concept of such importance in mathematics that they advocate it as a basis for organizing the instructional materials of the courses. Traditional algebra consists of a series of topics that are presented without indication of their relationships. To the teacher the relationships are known, but to the pupil they are not known. To him the changes from one topic to the next seem abrupt and bewildering. He is confused as to the real nature of algebra. He fails to see how the various chapters relate to one another and contribute to the final aims of the course. There is need for a principle which unifies the subject matter of algebra. It is believed by some that the function concept supplies this need. To illustrate, let one of the aims of first year algebra be the complete understanding of the function $y = ax + b$, as far as this may be possible on the ninth grade level. If a, x, and b assume all possible positive and negative values including zero, the laws of signs may be related to the problem of finding the values of $ax + b$ for given values of a, x, and b. Furthermore, substitutions and evaluations will show how y depends on x for its value and how y changes when x changes. If b is zero, y varies directly as x, which leads to the study of direct variation. By transforming $y = ax$ to $\frac{y}{x} = a$, one may introduce the study of ratio, the process of reduction of fractions, and some simple work with factoring. If fractional values of a are used, the subject of proportion may be touched upon. Tables are made by listing corresponding values of x and y. From the tables it is but a simple step to graphical representation. Of special interest is the case of $y = 0$, which is easily related to the solution of a linear equation in one unknown. Finally, there will be an abundance of problems leading to linear equations in one and in two unknowns. It is clear that practically all the traditional topics of algebra may be touched upon and that the work will be of a type simple enough for the beginner. At the

same time, the pupil will see that everything he studies has a purpose because the various topics are all related to a major aim, the understanding of $y = ax + b$.

The second stage of first year algebra will be a study of $y = ax^2 + bx + c$ similar to that outlined for $y = ax + b$. It is a review and extension of all the work offered in the first stage, and finally leads to quadratic equations in one unknown and to systems in two unknowns, one of which may be linear and the other quadratic. Functions of a degree higher than the second in one or in several variables may then complete the year's work.

In geometry the function concept may be used to organize subject matter by grouping together theorems dealing with relationships between arcs, central angles, and chords; angles whose sides touch or intersect a circle; segments of intersecting chords, tangents, and secants; measurement of surfaces; and regular polygons related to the circle. It is thus possible to group together bodies of closely related facts and principles which the pupil may study with clearly stated objectives before him. Such a course will appeal to the learner and will be better understood than one in which logic is the sole basis of organization.

IV. Constructing a Test to Measure Functional Thinking

Measuring growth in functional thinking. The first step in planning the program of measuring the development of functional thinking was the construction of an instrument of measurement. No published test was available for the purpose but some material was collected from mathematics textbooks. It was necessary to create additional suitable test material. The test items were classified under eight headings to test eight characteristics of functional thinking:

1. Recognizing relationships.
2. Understanding how a change of one variable in a formula affects the others.
3. Interpreting graphs.
4. Making changes in equations.
5. Recognizing linear and quadratic functions.
6. Understanding dependence in formulas.
7. Using the language of variation.
8. Expressing relationships in mathematics symbols.

The number of test items turned out to be too large for practical purposes since it seemed desirable to make a test that could be administered in one class period. Hence materials that seemed too difficult or of comparatively less importance were eliminated. The test was put in typewritten form and administered to pupils of various high school levels. As a result, the statements of some items were improved, some items were omitted, and a few items were added.

The test was then mimeographed and administered to 1,802 high school pupils, approximately 500 pupils of each high school grade. To secure uniform administration the following directions were sent to the teachers:

> The best way to give the test is to use a portion of each of two consecutive class periods, Parts I, II, and III the first period and the remainder the second. No time limits are set as it is intended to give every pupil time to try every item.
>
> If your classes are grouped according to ability, do not select the brightest or poorest sections for the test. What is needed is a representative group of pupils.
>
> Have classroom conditions as nearly normal as possible and give all directions in your usual manner.
>
> Distribute the test booklets, turning them face down. Say: "Do not open the booklets. Fill in the blank spaces for name, age, etc., but do not start to work on the test. The test contains some problems which you may not be able to work. Work the problems that you know and do not spend much time on the others. When you have finished the test, lay it aside and study your regular work." (The teacher may write an assignment on the board to keep those who finish early profitably employed.)

A copy of the test follows:

TESTING FUNCTIONAL THINKING

GRADES 9 TO 12

Name ______________________ Age ______________
Years Months

City ______________________ State ______________

School ______________________ Grade ______________

Teacher ______________________ Date ______________

Part I. Recognizing Relationships

One quantity often depends on another for its value. Thus, the enrollment in the public schools depends on the size of the population, and the price paid for wheat depends on the amount of rainfall. To show how well you recognize relationships write on each blank line the word that makes the sentence complete:

1. If a train travels at a uniform rate the distance depends on ——————————.
2. The amount of pay a man receives depends on the number of —————— he works and his —————— per day.
3. The perimeter of an equilateral triangle depends on the ——————.
4. The premium of a life insurance policy depends on —————— —————— of the applicant.
5. The interest received from an investment depends on the —————— and the ——————.
6. The time required by a pendulum to make one vibration depends on the ——————.
7. The amount of expansion due to heating depends on —————— ——————.
8. The area of a triangle depends on the —————— if the base remains the same.
9. The amount of daylight in a room depends on the —————— ——————.

Score for Part I = Number right = ——————

Part II. Ability to determine how a change of one variable in a formula affects the others.

A. The formula $d = rt$ gives the distance an automobile travels in any given time. If $r = 35$ and remains the same, and if t changes in value from 0 to 10, find d

1. when $t =$ 0, 3, 7, 9, 10,

Ans. $d =$ ——— ——— ——— ——— ———

If a train travels at a rate of 40 miles an hour you know that the relation between the distance and time is $d = 40t$. Furthermore,

if $t = 0$	3	5	6	9	10
then $d = 0$	120	200	240	360	400

If you examine the table you will see that as t in the formula $d = 40t$ changes, d also changes.

In the blank spaces write the changes of d corresponding to the following changes of t:

Changes of t	Corresponding changes of d
2. If t increases	d ——————————
3. If t decreases	d ——————————
4. If t is doubled	d ——————————
5. If t is divided by 2	d ——————————

Score for Part II, A = Number right = ——————

B. If in the formula $c = 3.4d$ the value of d changes, tell how c changes:

Changes in d	Changes in c
1. d is doubled	c ——————————
2. d is divided by 3	c ——————————
3. d is increased by 5	c ——————————
4. d is diminished by 4	c ——————————

Score for Part II, B = Number right = ——————

C. In the formula $s = \frac{1}{2}gt^2$ the value of t is made to change. Tell how the value of s changes:

Changes in t	Changes in s
1. t is multiplied by 2	s ______
2. t is divided by 4	s ______

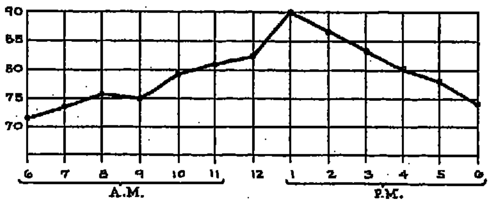
3. In the formula $v = abc$ how does v change if a and b are kept constant and c is doubled? ______
4. How does v change if a is kept constant, b is doubled, and c is multiplied by 3? ______

Score for Part II, C = Number right = ______

Total Score for Part II (A + B + C) = ______

Part III. Interpreting Graphs

A. The diagram below represents the changes in temperature during a certain day.



From the diagram answer the following questions:

1. When was it warmest? ______
2. When was it coolest? ______
3. When was the rise of temperature greatest?
   From ______ to ______
4. When was the change least?
   From ______ to ______
5. When was the temperature 80°? ______

Score for Part III, A = Number right = ______

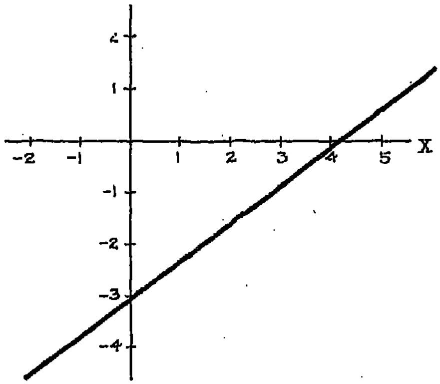
B. The graph on the following page represents the polynomial $\frac{3x}{4} - 3$.

1. Let x change as in the table below and write the corresponding values of $\frac{3x}{4} - 3$ in the empty spaces:

x	0	1	2	3	4	5	−1
$\frac{3x}{4} - 3$							

2. As x changes from 0 to 5 how does $\frac{3x}{4} - 3$ change? Ans. ____________

3. As x changes from 0 to -2 how does $\frac{3x}{4} - 3$ change? Ans. ____________



4. In the diagram draw a dotted line to represent the value of $\frac{3x}{4} - 3$ when $x = 2$.

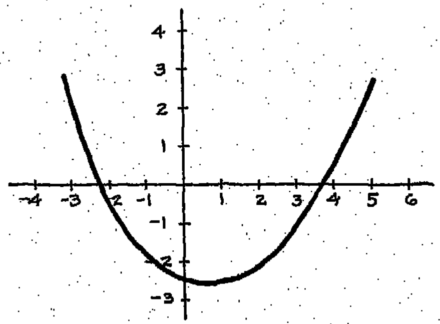
5. The value of $\frac{3x}{4} - 3$ is zero when $x =$ ____________

6. The solution of the equation $\frac{3x}{4} - 3 = 0$ is $x =$ ____________

Score for Part III, B = Number right = ____________

In most of the following problems it will be necessary to give approximate values:

C. 1. The graph shown below represents a function of the general form:



2. As x varies from 0 to 1 the function varies from ____________ to ____________

3. As x varies from 1 to 4 the function varies from ____________ to ____________

4. As x increases without bound the function ______________________

5. As x decreases from 0 to -3 the function ________ from ________ to ________.
6. The value of the function is zero when $x =$ ________.
7. Draw the axis of symmetry.
8. The minimum value of the function is ________

Score for Part III, C = Number right = ________

Total Score for Part III (A + B + C) = ________

Part IV. Making changes required in the solution of equations

To preserve the equality a change in one side of an equation often requires a corresponding change in the other. In the following examples change the right side, if necessary to preserve the equality. Otherwise write it as it is:

$$12x + 18 = 15x + 6$$

	Left side		Right side
1.	$18 + 12x$	=	________
2.	$6x + 9$	=	________
3.	$4x + 6$	=	________
4.	$24x + 36$	=	________
5.	$12x + 12$	=	________
6.	18	=	________
7.	$10x + 18 + 2x$	=	________
8.	$12x + 20$	=	________

Score for Part IV = Number right = ________

Part V. Recognizing linear and quadratic functions

Show that each of the equations 1 to 3 is of the form $ax = b$:

1. $5x = 60$ is of the form $ax = b$ since $a =$ ________, $b =$ ________
2. $2\pi r = 84$ is of the form $ax = b$ since $a =$ ________, $b =$ ________
3. $200 = .04p$ is of the form $ax = b$ since $a =$ ________, $b =$ ________

Show that each of the right members in Exercises 4 to 8 is of the form $ax^2 + bx + c$ by giving the values of a, b, and c:

4. In $y = 3x^2 + 5x - 1$ we have $a =$ ________, $b =$ ________, $c =$ ________
5. In $y = 2x^2 - 5$ we have $a =$ ________, $b =$ ________, $c =$ ________
6. In $s = \frac{1}{2}gt^2$ we have $a =$ ________, $b =$ ________, $c =$ ________
7. In $c = \pi r^2$ we have $a =$ ________, $b =$ ________, $c =$ ________
8. In $s = s_0 + \frac{1}{2}gt^2$ we have $a =$ ________, $b =$ ________, $c =$ ________

Score for Part V = Number right = ________

Part VI. Understanding dependence in formulas

Make the following sentences complete:

1. $S = (n - 2)\,180$ The sum of the angles of a polygon depends on ________.

2. $A = \frac{a^2}{4}\sqrt{3}$ The area of an equilateral triangle depends on ______.

3. $x = \frac{1}{2}x'$ The angle inscribed in a circle depends on ______.

4. $A = \frac{1}{2}bh$ The area of a triangle depends on ______.

5. $a = 180 - (b + c)$ One angle of a triangle depends on ______.

6. $i = .04pt$ The income received from an investment depends ______.

7. $y = \sin x$ The value of the sine function depends on ______.

Score for Part VI = Number right = ______

Part VII. Variation

State in symbols:

1. y varies directly as x: ______
2. y varies inversely as x: ______
3. The price of sugar varies as the weight: ______
4. The weight of steel wire varies as the length: ______
5. The distance an object falls varies as the square of the time: ______
6. The area of a rectangle varies as the base and altitude: ______
7. x varies inversely as y: ______
8. The time of traveling a fixed distance varies inversely as the rate: ______

Express in the language of the variation the following formulas:

9. $C = 2\pi r$: ______
10. $i = .05p$: ______
11. $M = dV$: ______
12. $IR = E$: ______
13. $S = \frac{1}{2}gt^2$: ______
14. $PV = K$: ______

Express as an equation the relationships between x and y which are illustrated in the following tables:

15.

x	2	3	4	5	6
y	6	9	12	15	18

Relationship: ______

16.

x	1	2	3	4	5
y	12	6	4	3	2.4

Relationship: ______

Score for Part VII = Number right = ______

Part VIII. Expressing relationships in mathematical symbols

For each of the following variables write on the blank line in algebraic symbols the formula expressing the relationship:

Variables	Relationships
1. Area of square and side	____________
2. Cost of railroad ticket and distance of trip at 3 cents a mile	____________
3. Area of circle and radius	____________
4. Time and distance of falling object	____________
5. Centigrade and Fahrenheit	____________
6. Pressure and volume of gas	____________

Express by means of an equation each of the following relationships in two symbols, as x and y:

7. The speed of a train exceeds that of an automobile by 10 miles:	____________
8. John is 5 years younger than Henry:	____________
9. The area has been increased by 20 square feet:	____________
10. Paul's father earns 3 times as much as he:	____________

Express as an equation the relationship shown in the following table:

11.

x	1	2	3	4	5
y	3	5	7	9	11

Relationship: ____________

Score for Part VIII = Number right = ________

Total Score for Test = ________

To facilitate scoring, a set of answers was sent to the coöperating teachers with the request to return them unscored if they could not spare the time for marking. All tests were then scored by the writer to insure uniformity. The score sheet follows:

ANSWERS FOR THE TEST ON FUNCTIONAL THINKING

(Please return papers scored or unscored)
(In marking put a check mark to the right of correct responses)

Page 104

Part I

1. the time, or time and rate
2. hours, pay, or salary
3. side, sides, or base
4. the age
5. rate, time, or rate and time, or time
6. length
7. the temperature, or heat
8. altitude, or height
9. lighting surface, or window space

Page 104

Part II A

1. 0, 105, 245, 315, 350 (Counts 5 points)
2. increases
3. decreases
4. is doubled
5. is divided by 2

Page 104

Part II B

1. is doubled
2. is divided by 3
3. is increased by 5 x 3.4, or 17
4. is diminished by 4 x 3.4, or 13.6

Page 105

Part II C

1. is multiplied by 4
2. is divided by 16
3. v is doubled
4. v is multiplied by 6

Page 105

Part III A

1. 1 P.M.
2. 6 A.M.
3. from 12 to 1
4. from 8 to 9
5. at 10, or 4

Pages 105-106

Part III B

1. -3, $-2\frac{1}{4}$, $-1\frac{1}{2}$, $-\frac{3}{4}$, 0, $\frac{3}{4}$, $-3\frac{3}{4}$ (Counts 7 points)
2. from -3 to 0
3. from -3 to $-4\frac{1}{2}$
4. from 2 downward to the graph
5. 4
6. 4

Pages 106-107

Part III C

1. $ax^2 + bx + c$
2. from $-2\frac{1}{2}$ to $-2\frac{3}{4}$ (a close approximate value will do)
3. from $-2\frac{3}{4}$ to $\frac{1}{4}$ (a close approximate value will do)
4. increases without bound
5. increases from $-2\frac{1}{4}$ to $+1$ (accept approximate values)
6. $3\frac{3}{4}$, $-2\frac{1}{4}$ (accept approximate values)
7. a line from 1 downward to the curve
8. $-2\frac{3}{4}$ (accept approximate value)

Page 107

Part IV

1. $15x + 6$
2. $\dfrac{15x + 6}{2}$ or $9x - 3$
3. $5x + 2$ or $7x - 6$
4. $30x + 12$ or $27x + 24$
5. $15x$
6. $3x + 6$
7. $15x + 6$
8. $15x + 8$

Page 107

Part V

1. 5, 60
2. 2π, 84
3. .04, 200
4. 3, 5, -1
5. 2, 0, -5
6. $\frac{1}{2}g$, 0, 0
7. π, 0, 0
8. $\frac{1}{2}g$, 0, 80

Pages 107-108

Part VI

1. the number of sides, or number of angles
2. the length of the side
3. the intercepted arc
4. the base and altitude
5. the sum of the other two or the other two
6. the principal and time or principal, rate, and time
7. the angle

Page 108

Part VII

1. $y = cx$ or $\dfrac{y}{x} = c$
2. $xy = c$
3. $p = cw$
4. $w = cl$
5. $d = ct^2$
6. $A = cbh$
7. $xy = c$
8. $rt = c$
9. circumference varies as radius
10. interest varies as investment
11. mass varies as volume
12. current varies inversely as the resistance
13. distance varies as square of time
14. pressure varies inversely as the volume
15. $y = 3x$, or $x = \frac{1}{3}y$
16. $xy = 12$

Page 109

Part VIII

1. $A = s^2$
2. $c = 3n$
3. $A = \pi r^2$
4. $s = \frac{1}{2}gt^2$
5. $F = \frac{9}{5}C + 32$
6. $pv = c$
7. $x = y + 10$, or $v_1 = v_2 + 10$, or $x - y = 10$, or $v_1 - v_2 = 10$
8. $x = y - 5$, or $a_1 = a_2 - 5$, or $a_1 + 5 = a_2$, or $x + 5 = y$
9. $x = y + 20$ or $A_2 = A_1 + 20$
10. $x = 3y$, or $3y$
11. $y = 2x + 1$

The results of the test, when computed and interpreted, should give a fair picture of the extent to which functional thinking is being developed among pupils of the various levels of the high schools, in so far as the pupils and schools are representative and as functional thinking is measured by the test. However, the investigation is not to be limited to the 1,802 pupils who have been tested so far. The next step will be to give the test to the pupils of Grades IX to XII in the laboratory school of the University of Chicago, and in several coöperating schools. A year later it will be given again in the same schools to determine the extent to which progress has been made by individual pupils. Most likely it will be given once more a year from that date. Test results will be discussed in relation to the mental ability of the pupils and to achievement in school subjects.

Copies of the tests were sent to the various schools on September 1 and returns came in promptly. However, on account of the limited time, it was impossible to wait until all the tests had been returned. Hence only one-half the papers sent out, that is, 901 papers, were included in this report. The present report should be regarded as preliminary, and a more extensive report on all the tests returned will be published at a later time.

V. Discussion of the Findings of the Study

Ability to state changes when they are illustrated graphically. It is not claimed that the test is a perfect instrument of measuring the pupil's ability to think functionally. However, the test items cover those phases of the function concept that were disclosed by an examination of some of the best textbooks and are most likely to receive attention in the teaching of mathematics.

The best results were obtained with the first part of Test III, Interpreting Graphs. The highest possible score on the test is 5 and the medians for the various levels of the high school were all either 4 or 5. In all levels, however, except XIA and XIIA there were some pupils who made a zero score. It seems, therefore, fair to say that freshmen just entering the high school are able to express the changes illustrated in such simple graphs as the daily temperature graph and that they retain this ability throughout the secondary school period. There is a tendency for them to gain in this ability as they continue in school, although mastery is not always attained.

The results obtained in the test on recognizing changes in graphs of mathematical laws are not nearly as encouraging as those for the statistical graph. The second and third parts of Test III contain problems relating to the changes illustrated in the straight line graph of $\frac{3x}{4} - 3$ and in a parabola representing the function $ax^2 + bx + c$. The highest possible score on the straight line graph test is 12. The medians for Grades IX, X, and XIB are all zero, for XIA and XIIB they are 6, and for XIIA the median score is 8. It seems, therefore, that the ability to interpret the straight line graph is not developed until the pupil has finished the first semester of his junior year. However, in all levels of the school, pupils were found who were unable to answer correctly any of the 12 questions in the test. On the other hand, some of the freshmen, although just beginning secondary school mathematics, and some sophomores, were able to answer correctly as many as nine of the twelve questions. Several juniors and seniors made perfect scores.

The median scores on the last part of Test III, changes in the parabola, were zero for all classes. Some pupils, however, in the upper classes responded correctly to as many as 8 out of the 12 questions.

It was noticed that when pupils answered problems as to changes of $\frac{3x}{4} - 3$ they invariably obtained their information from the table rather than from the graph, which indicates that they understand the tabular representation of a function much better than the graphical. Further evidence that the graph is not thoroughly understood is found in the scarcity of responses to the problem: In the diagram draw a dotted line to represent the value of $\frac{3x}{4} - 3$ when $x = 2$. Better results were obtained with the problem: The value of $\frac{3x}{4} - 3$ is zero when $x =$ ——. It seems that the correct answers to this problem may be explained as the result of good memory rather than understanding of the graph.

One item in Test IIIB calls for evaluation of $\frac{3x}{4} - 3$ for $x = 0, 1, 2, 3, 4, 5, -1$. The ability to do this problem successfully does not appear until the first semester of the junior year has been finished, although throughout first-year algebra more

difficult exercises are given in all textbooks. Errors which commonly occur are:

$\frac{3x}{4}$ for $x = 0$ is $\frac{3}{4}$ $\qquad$ $\frac{3x}{4}$ for $x = -1$ is $\frac{3}{4}$

$\frac{3x}{4}$ for $x = 2$ is $\frac{6x}{4}$ $\qquad$ $\frac{3x}{4}$ for $x = 2$ is $\frac{5}{8}$

and similar mistakes for other values of x.

The same type of problem occurs in a simpler form in Test IIA. Here the pupil is to evaluate $35t$ for $t = 0, 3, 7, 9, 10$. Very little difficulty was experienced. The one noticeable error was caused by the zero. The value of $35t$ for $t = 0$ was frequently given as 35 instead of zero. It is surprising that in the advanced classes pupils should continue to find it so much more difficult to evaluate $\frac{3x}{4}$ than $35t$.

Attention was called previously to the tendency of pupils to prefer to study changes in functions by use of the tabular rather than the graphical representation. Part IIA of the test contains four items relating to changes to be discovered from a table representing $d = 40t$. No particular difficulty was encountered, the median on the four items being 4 for all classes. In IIB the changes in the formula $c = 3.4d$ were to be found, but no table was given. This proved to be far more difficult. The medians vary from 0 to 2 from the freshman to the senior year. Very few pupils could answer more than the first two questions. In IIC the changes in $s = \frac{1}{2}gt^2$ and $v = abc$ were to be found. In all classes the median was zero, but in every group some pupils were found who could answer three or all of the four questions. Changes in $s = \frac{1}{2}gt^2$ and $v = abc$ are far more difficult to understand than changes in $d = 3.4t$, which in turn is more difficult than $d = 40t$. Very few pupils seem to acquire during the high school period anything like complete understanding of the relationships involved in functions like $s = \frac{1}{2}gt^2$ and $v = abc$.

An interesting frequent error appeared in question 4 of IIC: If in $v = abc$, a is constant, b doubled, and c multiplied by 3, how does v change? Nearly as many pupils gave the answer 5 as gave 6.

Recognizing dependence of one variable on others. Parts I and VI were designed to measure the ability to recognize dependence of one variable on others. In I the relationships are expressed in

verbal form and in VI they appear in the form of well-known formulas. It was found that some pupils enter the high school with considerable ability to recognize dependence when it is expressed verbally. Some were able to respond correctly to eight out of the nine questions. The median was 4 for the freshmen and sophomores, 5 for those beginning third-year work, and 6 for the remaining pupils. Many pupils do not think quantitatively. They give in all seriousness answers like the following:

> The time required by a pendulum to make one vibration depends on *the clock.*
>
> The amount of expansion due to heating depends on *the weather, furnace, or coal.*
>
> The premium of a life insurance policy depends on *the health, signature of the applicant.*
>
> If a train travels at a uniform rate the distance depends on *the engineer.*

Apparently these pupils fail to see quantitative relationships in the questions.

Dependence is far more difficult to recognize in formulas than in verbal statements. The only formula of the seven given in the test that seems to be familiar to pupils of all levels is $A = \frac{1}{2}bh$. The formula $i = 0.04pt$ offered considerable difficulty.

The median for the freshman group is zero. It increases by 1 each year, reaching 3 in the senior year. A few pupils who gave correct responses to all seven questions were found in Grade XA and Grade XIIB. It seems that pupils in general do not think of dependence or relationships when they are dealing with formulas.

Part VIII contains six items which call for statements of well-known formulas relating given facts, and five items calling for symbolic statements of relationships stated verbally. There is evidence that this phase of functional thinking is not usually acquired by pupils before the middle of the junior year, although there are a few in all classes who respond correctly to from 7 to 10 of the eleven questions. In all classes up to and including the first semester of the junior year one-half the pupils were unable to answer correctly a single question.

Recognizing relationships in equations. In solving an equation the pupil has to make certain changes to obtain the required value of the unknown. A change made in one member of the equation must be followed by a corresponding change in the other if the equality is to be preserved. Part IV aims to measure the ability

to recognize the changes made in one member and to make the corresponding changes in the other. Eight items were selected for this purpose. The results indicate that the ability is not generally acquired before the middle of the junior year. The medians for the preceding years are either zero or one. However, as in all tests in every group, there are individuals who responded correctly to all eight questions.

In view of the fact that opportunities for using equations occur almost continually in all mathematical courses the lack of ability shown by the test is surprising. Apparently, relationships are not used in solving equations and mechanical performance is substituted for functional thinking. Undoubtedly the underlying principles have been explained at the beginning but they are not retained because they are soon discarded in favor of some purely mechanical device, such as transposition or cancellation. The opportunity for training in functional thinking is thereby lost.

Recognizing linear and quadratic functions. The first three items of Part V aim to measure the ability to recognize such equations as $5x = 60$, $2\pi r = 84$, and $200 = 0.04p$ as special cases of the general equation $ax = b$. The pupil is required to state the values of a and b. Five more test items were selected to determine a similar ability with $ax^2 + bx + c$ by naming a, b, and c. The results show that the ability is acquired only by outstanding individuals. The median of the highest group was 3. The medians for the freshman and sophomore groups was zero. Part V turned out to be one of the most difficult parts of the test.

Variation. This phase of functional thinking is not usually touched upon before the junior year. However, as far down as in Grade IXB some pupils were able to respond correctly to five of the sixteen questions.

All the groups possess little knowledge of the language of variation. The best showing is made by Grades XIA and XIIA but even for them the median is only 1. The highest score—12—is made by a pupil in Grade XIIB.

VI. Conclusions

The foregoing report is concerned with the first step in an investigation aiming to measure the development of functional thinking. It has been shown that understanding of nearly all characteristics of functional thinking is acquired early by individual

MEDIANS AND RANGES FOR THE VARIOUS GRADES FOR ALL PARTS OF THE TEST

Classes		IXB	IXA	XB	XA	XIB	XIA	XIIB	XIIA
No. of pupils		203	61	196	69	141	50	167	14
Part I Highest score 9	Median Range	4 3–8	4 0–8	4 0–9	4 0–7	5 1–9	6 2–8	6 2–10	6 3–8
Part II A Highest score 10	M R	7 0–9	6 0–9	7 0–9	8 0–9	8 0–9	9 3–9	9 0–9	8 4–9
Part II B Highest score 4	M R	0 0–4	1 0–4	1 0–4	2 0–4	2 0–4	2 0–4	2 0–4	2 0–4
Part II C Highest score 4	M R	0 0–3	0 0–3	0 0–4	0 0–2	0 0–4	0 0–4	0 0–4	0 0–4
II A + B + C Highest score 17	M R	6 0–14	8 0–16	8 0–15	9 0–15	10 0–17	11 3–17	11 0–17	10 5–17
Part III A Highest score 5	M R	4 0–5	4 0–5	4 0–5	5 0–5	5 0–5	5 1–5	4 0–5	5 4–5
Part III B Highest score 12	M R	0 0–9	0 0–9	0 0–11	0 0–9	0 0–12	6 0–11	6 0–12	8 0–11
Part III C Highest score 8	M R	0 0–3	0 0–2	0 0–5	0 0–2	0 0–8	0 0–6	0 0–8	0 0–4
Part III A+B+C Highest score 25	M R	5 0–14	5 0–13	5 0–20	5 0–14	5 0–25	12 2–20	11 0–24	12 4–20
Part IV Highest score 8	M R	0 0–8	1 0–7	1 0–8	1 0–8	1 0–8	5 0–8	1 0–8	7 0–8
Part V Highest score 8	M R	0 0–6	0 0–5	0 0–8	0 0–4	1 0–7	3 0–8	2 0–8	2 0–5
Part VI Highest score 7	M R	0 0–4	0 0–4	1 0–4	2 0–7	2 0–6	3 0–6	3 0–7	3 0–5
Part VII Highest score 16	M R	0 0–5	0 0–2	0 0–6	0 0–2	0 0–11	1 0–9	0 0–12	1 0–5
Part VIII Highest score 11	M R	0 0–9	1 0–7	0 0–7	0 0–6	0 0–7	4 0–10	3 0–10	5 0–8

pupils. With other pupils they develop slowly, and many never acquire them at all. The best results are obtained in the tests on interpreting changes in the statistical graph, recognizing dependence when relationships are presented in verbal statements, and recognizing changes in tables of numerical facts. Pupils bring to the secondary school considerable knowledge about these matters and are able to retain and extend this knowledge as they continue in school work.

For training in all other types of functional thinking the pupil depends entirely on the secondary school. In some characteristics little or no development is noticed before the pupil finishes the first semester of his junior year. This applies to the ability to recognize changes in equations, to understand relationships in line graphs and in formulas, and to know the meaning of linear and quadratic functions.

The remaining characteristics are not acquired by groups to any appreciable extent, but they are attained by outstanding individuals. There were no items on the entire test to which no correct responses were made by some pupils. On the other hand, no pupils were found who responded correctly to all the items in the test.

Bibliography

AUSTIN, EDNA E. "The Function Concept and the Tenth Year Pupil." *Bulletin of Kansas Association of Mathematics Teachers*, No. 31, pp. 5-8. Manhattan, Kansas.

BLANK, LAURA. "Variability and Functionality in High School Mathematics." *Mathematics Teacher*, XXII (November, 1929), 405-12.

BRESLICH, E. R. *Problems in Teaching Secondary School Mathematics.* Chap. IX. Chicago, The University of Chicago Press, 1931.

CHRISTOFFERSON, H. E. "The Graph as a Means of Picturing Relationships." *Mathematics Teacher*, XXI (April, 1928), 227-28.

EVERETT, J. P. *The Fundamental Skills of Algebra*, pp. 77-79. New York, Bureau of Publications, Teachers College, 1928.

GEORGES, J. S. "Functional Thinking as an Objective of Mathematical Education." *School Science and Mathematics*, XXIX (May, June, 1929), 508-17, 601-08.

HASSLER, J. O., and SMITH, R. R. *The Teaching of Secondary Mathematics.* Chap. X. New York, The Macmillan Company, 1930.

HEDRICK, E. R. "Functionality in Mathematical Instruction in Schools and Colleges." *Mathematics Teacher*, XV (April, 1922), 191-207.

Hennessey, F. P. "The Principle of Continuity." *Mathematics Teacher*, XXIV (January, 1921), 32-40.

Schaaf, W. L. *Mathematics for Junior High School Teachers.* Chap. XVI. Richmond, Johnson Publishing Company, 1931.

MATHEMATICS AND MEASURING OF WORLD TRENDS AND FORCES, A.D. 1932

By W. S. SCHLAUCH
New York University, New York City

Introduction

Need of quantitative knowledge. The achievements of mathematics when applied to celestial mechanics have long been celebrated. The accuracy with which astronomers are able to predict the coming and duration of an eclipse, the conjunction or opposition of planets, or the return of a comet which is following a closed path, is so well known that even those who read only the daily newspaper know that in this field at least the coming event can be mathematically calculated and foretold. Yet the masses of mankind continue to regard economic, social, and political changes as fitful, arbitrary, and unpredictable. Although the law of cyclone development and motion, and the general theory of wind circulation on the earth, have been mathematically investigated since Ferrell's time, it is still not uncommon to hear of the destruction following a cyclone's advent spoken of as "an act of God," with the distinct implication that it is a mysterious visitation, not predictable or subject to law.

Although the need of quantitative knowledge and the ability to predict has been acute for a long time, only recently has serious work been undertaken by mathematically trained economists and social scientists to supply this lack. The widespread unemployment, depression, and dislocation of our industrial and economic life are forcing us to try to estimate and measure the trends and forces operating in the world to-day, and to try to modify and control them, in part at least, so as to avoid the recurrence of such acute world agony as we are now experiencing. This chapter will attempt to outline briefly the major fields now being investigated by mathematical methods—largely mathematical statistics—and to indicate by concrete illustrations the type of mathematical method used.

Although the limitation of space allows only a sketchy and inadequate treatment an attempt will be made to present in outline:

I. The causes of world transformation to the modern industrial era.

II. Present trends and forces in—

1. The economic field
2. The social field
3. Intellectual and educational fields.

Let me warn the reader in advance that the interpretations and conclusions here advanced are tentative. In fields involving such vast masses of facts and phenomena and with the sciences dealing with these fields still largely descriptive, no man can hope to do more than to indicate some of the major operating forces, and the quantitative methods that are being employed, and to give such conclusions as are reasonable from the facts and principles ascertained. And it is essential that the reader maintain something of the scientific attitude of mind which welcomes revision and enlargement of the stock of ideas, and hugs no superstition or fixed idea so closely that it can not be surrendered in the face of conclusive evidence.

I. World Transformation Causes

The hand-tool ages. The modern world presents the picture of a machine civilization. The vast multiplication of man's power by harnessing the forces of nature through the use of the machine has made it possible to feed, clothe, and house the great masses of people living in the world to-day. The ancient and medieval times were alike in being hand-tool ages.

If the Martians had sent an expedition to visit the earth in the time of Egypt's pyramid dynasties, they would have found a civilization based on *slavery, domestic economy,* and *barter.* The work of the world was performed by hand, with simple tools, and the mass of the population were slaves—living machines—in order that civilization might be carried on by the favored few. To the Pharaoh and the priests, the cry of the slaves was "mellowed by distance into monotonous music." Our Martian scientists would probably have said: "A real civilization is not possible with such primitive tools, and so general an ignorance of science. We will send an expedition to Earth again in a few thousand years."

If they had returned to Earth about 1690, they would still have found men using hand looms, mattocks, hand tools, and man power. But they would have found a fair security in the miserable life of the artisan. The town money economy was coupled with a local market which was steady and the worker owned his own tools. He kept the proceeds of his labor, except those which were taken from him by his lord or seignior. Exploitation was direct and based on aristocratic privilege. Our Martians would doubtless have shaken their heads at the long delay in the coming of science, invention, the harnessing of nature by machinery.

Beginning of the Industrial Revolution. But the change began about 1738. In that year Kay's flying shuttle was perfected and soon after this Hargrave and Arkwright perfected spinning machinery. Cartwright invented the power loom in 1784. Water power was now harnessed, and steam power and transportation *were developed between 1750 and 1830. The change initiated has* been called the Industrial Revolution, because industry, as we know it, became possible with the advent of the machine.

Now, the establishment of large factories with expensive machines was possible only for those who had great possessions. So we see the worker separated from ownership of the tools with which he worked, gathered with other workers in factories and working on such terms and for such remuneration as the owner or capitalist dictated. The history of the Industrial Revolution in England reveals the terrible exploitation of the worker in the transition period, before the factory legislation of the nineteenth century brought him some relief. The worker's wages were fixed so low that his purchasing power was inadequate to absorb the flood of goods turned out by the factories. The need of foreign markets for the surplus thus arose, and England built up her Empire to acquire exclusive markets for her surplus manufactures. Her navigation acts, laws forbidding manufactures in the American Colonies, and oppressive trade regulations, all designed to secure a market for her surplus goods, cost her her thirteen American Colonies.

With the spread of invention and the resulting development of industry, the capitalists or owning class acquired increasing political power. Parliaments were strengthened throughout the western world, and the "burgers" or bourgeois enterprisers wrested power from the old feudal aristocracy and consolidated their position as an owning class. At the opening of the twentieth century the

modern world in Europe and America may be briefly described as an industrial machine civilization with the capitalist enterpriser class holding the substance of economic and political power, absorbing a large share of the product of industry; a large mass of laboring or working population dependent on such wages as they could command from the owning class; and the places of former power filled by the remnant of the feudal aristocracy in Europe, but practically devoid of power.

By that time the industrialization of all the great European nations, as well as of the United States, had taken place, and the surplus goods which could not be absorbed by the home population had driven all these governments into the race of imperialism. England's far-flung empire was copied on a smaller scale by the French North African Empire, Germany's East African and Pacific possessions, Italy's Tripolitan and Abyssinian holdings, Belgium's Congo "Free" State with its horrible oppression of the native population, and the Caribbean and Philippine protectorates of the United States.

The present-day situation. Each of these great powers was and is determined to keep its territorial possessions for exclusive exploitation and marketing. So we have the spectacle of vast armies and navies, the holocaust of 1914-1918, and the same military and naval burdens since the World War. Imperialism exercised for exclusive markets and exploitation always means excessive nationalism, rivalry, jealousy, and preparation for war. Leaving this brief and inadequate sketch of the rise and nature of the modern world, we turn now to a consideration of the second point.

II. Present Trends and Forces

In the economic field. (A) *Mass production and the giant integration of industry.* Most articles in common demand to-day are machine made and produced in large numbers, with the process minutely subdivided and standardized. This is known as mass production, and it cuts down the cost of production per unit. Because many machines are employed to produce the parts, the labor cost per unit is greatly reduced and the profits to the producer are increased. Automobiles, iron pipe, structural steel, sewing machines, plows, textiles, machinery, furniture, and flour are all so produced. Anyone who has inspected a Ford assembly plant and has watched the assembling of a completed car as the endless

belt traveled through the plant, each worker adding his piece, tightening a bolt, or adjusting a part, realizes that the tempo of modern mass production is very fast.

Mass production carries with it the tendency to create giant corporations, to make this large scale production possible. A typical example is the United States Steel Corporation, a holding company which unites under one control ten major corporations, with subsidiaries and sub-subsidiaries, making in all more than one hundred and seventy component companies. They control and own ore mines, coal mines, fleets of vessels, smelting and reduction plants, foundries, factories, and finishing plants. Their operations cover the earth, and their capitalization exceeds a billion dollars.

One inevitable result of this organization of industry is that the worker is at a disadvantage in bargaining for employment. For there remain few employers to whom he can offer his services in any line, and he is feeble indeed compared with the tremendous power of the employing corporation. The desire to overcome this disadvantage has led to the attempt to form unions and bargain collectively. In the United States the steel workers have not succeeded in forming a union, nor are the coal miners' unions strong enough to meet the employers on equal terms, so that wages in these industries are low, and the purchasing power of the employees is not great enough to provide a sustained market for the stream of goods consumer-produced in the United States.

Now this type of industrial organization with mass production and increasing machine efficiency calls loudly for the application of economic knowledge, statistical surveys, and a central regulating agency of some kind to guarantee that:

(1) The purchasing power of the mass of the people is maintained, so that the market will continue to absorb the goods produced, and

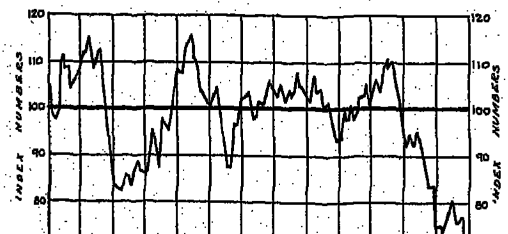
(2) That the trend of demand for the various kinds of goods may be estimated, so that an excess of one kind of goods shall not be produced at a loss while there is a deficit of other kinds of goods.

(B) *The business cycle and insecurity.* The lack of this balance and adjustment in distributing the proceeds of industry, and in adjusting production to demand, results in the business cycle and in insecurity.

The fact that business swings through the phases of depression, recovery, prosperity, and decline or collapse, repeating these four phases over and over again, is well known to all economists. Anyone who has read Professor Moore's *Economic Cycles, Their Law and Cause* or Professor Mitchell's *Business Cycles* knows that such cycles exist, that the phenomena involved are exceedingly complex and, under the present form of economic organization, very difficult to control. Index numbers, statistical theory, mathematical methods, such as least squares and the application of harmonic series, must be commonplaces to anyone who would investigate this field. Various economic services make estimates of the probable movement of the factors involved and sell their reports to business men who are interested.

We are now in the trough of a depression and the forces that are most active and have been for the past two years are eliminating inefficient and high cost producers, stimulating thrift and service on the part of those who have jobs, liquidating unsound ventures, and preparing the ground for recovery.

Uniting many barometers of business activity into a single index, we get a graphic picture of the movement of business. The graph of the index of business activity published by the *New York Times Annalist*, which the writer considers very reliable, is here reproduced. It is a weighted, average index, composed of ten separate indices of activity in the most important lines of production and distribution.



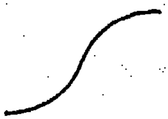
THE *Annalist* INDEX OF BUSINESS ACTIVITY

In interpreting the graph of the index of business activity, it must be borne in mind that a volume of activity which would yield an index of 100 in 1919 would yield an index *less* than 100

in 1929. The line marked 100 on the preceding graph, represents the estimate of the statistician of what is *normal* for the time indicated.

By studying the correlation of the movements of the various barometers of business conditions with one another and with the business index itself, various economic services try to forecast the movements of business activity, and to warn business men and industrialists of coming phases in the cycle. This problem is being attacked mathematically, and an encouraging beginning has already been made. It is evident that as business activity in general rises and falls, the market for well-established lines of consumers' goods will rise and fall, necessities experiencing a lesser range of variation than luxuries experience.

By reducing his output as the market demand declines, a producer can, of course, avoid overstocking with goods produced at relatively high cost, in the face of a falling price level. Thus, being able to adjust his business to the changing business cycle he will decrease losses and multiply gains. Sometimes, however, it is necessary to understand the play of additional forces. When a new industry, such as the automobile industry, results from invention, the growth of that industry tends to follow a law of growth which when graphed results in a line of growth such as is here shown. While the new invention is first winning its way, the growth is slow. When the public have been "sold" on the article, the slope of the curve of rate of increase is steep. When the public have absorbed practically all that they will purchase with a given standard of living and business activity, the curve begins to round toward its upper gradual slope, and additional growth is slow and dependent in the main on cheapening of the article and increase of population. In the steep part of the curve, the business cycle has less effect on the growth of the industry than in the earlier or later stages. The automobile industry in the United States is now leaving behind it the era of rapid expansion, and will be affected more and more by major business declines.



NORMAL GROWTH CURVE

(c) *Predicting the market for automobiles.* To describe in full detail how the prediction of the market for automobiles is attempted would require a book on the subject. One concrete phase of the process will be indicated. If a motor car manufacturer de-

sired to forecast the probable sales of his make of passenger car in the United States, he should forecast:

(1) The growth of the total of registered cars in the United States.
(2) The trend of the percentage of the total representing new cars and not re-registrations.
(3) The trend of his percentage of the total of new cars registered, year by year.
(4) The bearing of contemplated improvements and price reductions of his output on his position in the market, and such other relevant factors.

Some idea of the way in which mathematics is employed in making such forecasts can be gathered from the following study of the growth of the total number of registered passenger cars in the United States 1911-1925, inclusive, and its extrapolation to predict the numbers for 1926-1930, inclusive.

First, the data for total registrations were assembled by years, and various tests were applied to ascertain empirically the type of curve represented by the data. The conclusion was reached that the curve

$$Y = a\,b^{c^x}$$

was the best for the purpose. Applying logarithms we have:

$$\log y = \log a + c^x \log b.$$

Applying first differences,

$$\log y + \Delta \log y = \log a + c^{x+\Delta x} \log b.$$
$$\therefore \Delta \log y = \log b(c^{\Delta x} - 1)c^x$$
$$\therefore \log \Delta \log y = x \log c + \log k$$
$$\text{Where } k = \log b(c^{\Delta x} - 1)$$

This equation is linear in x and *log* Δ *log* y, and we can thus determine log c and log k by applying the equations of condition to determine the line of least square deviations:

$$m\Sigma x^2 + c\Sigma x = \Sigma xy$$
$$m\Sigma x + nc = \Sigma y$$

where *log* Δ *log* y is the y of the above equation, and *log* k is c, x is x.

Applying this method to the data, we found:

$$c = 0.926792, \text{ or } \log c = 9.966982 - 10;$$
$$\log k = 9.267082, \log a = 4.973466, \text{ and}$$
$$\log y = 4.973466 + 0.926792^x$$

from which values of log y and therefore y could be calculated for the years 1911----- where 1911 is the first year, and therefore $x = 1$. For 1912, $x = 2$-----; for 1913, $x = 3$; for 1914, $x = 4$.

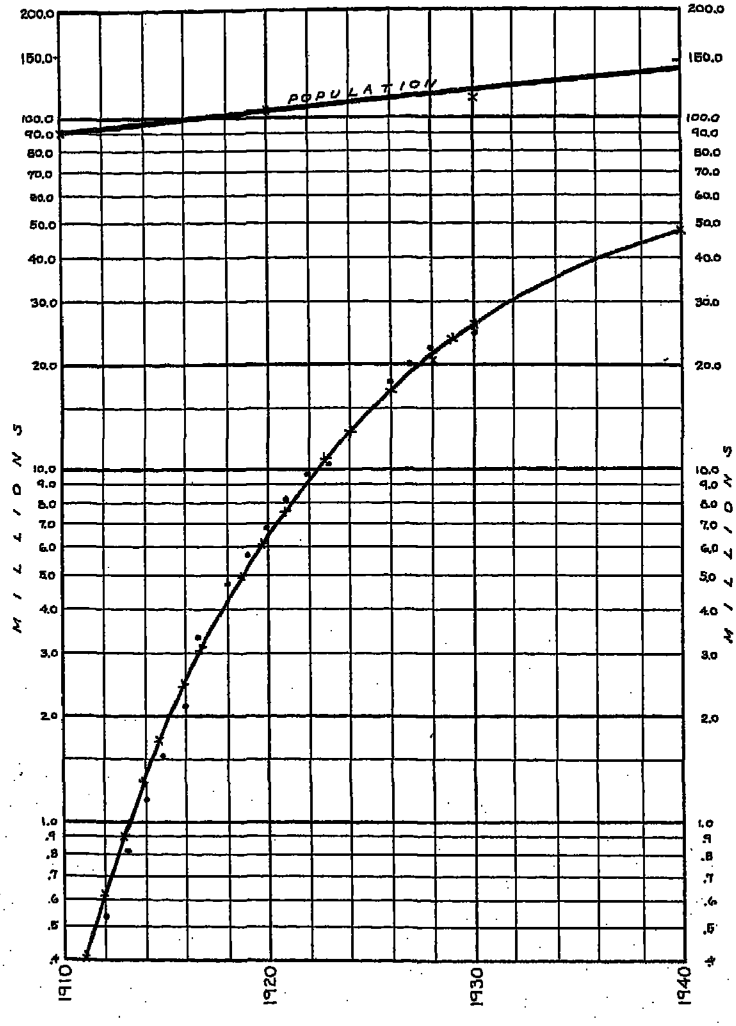
The results were plotted on arith-log paper, and the curve was produced beyond 1925 to determine the most probable total registrations for 1926-1930, inclusive. How closely this prediction agreed with the facts later developed may be judged from the graph shown on the following page.

It will be noted that in the graph the points of the curve of trend for the years are indicated by a "×," while the actual figures for the year are indicated by a "·". The equation of the curve was derived from the data for 1911-1925, but it will be noticed that the curve has been projected forward by means of its equation to 1940. The actual figures for 1926, 1927, and 1928 slightly exceed the predicted figures, as is shown on the graph, while those for 1929 and 1930 fall below the predicted totals. This is evidently the result of the fact that we were in a prosperity phase of the business cycle in 1926-1928, and that the cycle turned downward in the latter half of 1929.

But the trend of the numbers of passenger automobiles in the United States is clearly indicated. The population trend derived by similar methods is shown on the same graph. If no serious new factors enter, such as rapid development of air or flying machines, it seems probable that by 1940 there will be in the United States a population of 149,000,000, who will have registered passenger automobiles numbering approximately 49,000,000. Even now concrete roads are cutting up our countryside, and "double-decker" streets are being built in our large cities. It seems that the megalopolis of the not distant future in our country will have streets with four levels, pyramidal office buildings and dwellings that tower toward heaven, and a lock-step-regulated congestion which the people of to-day can scarcely conceive.

In the social field. All the factors of any civilization interact and are a complex whole. Economic changes unlock social changes as well as political changes, and these in turn initiate additional

economic changes. As an example, let us consider the status of woman, and marriage and divorce.



REGISTERED PASSENGER AUTOMOBILES AND POPULATION OF THE UNITED STATES

In any militant society, since fighting is the important business, woman is always man's chattel. And so we find in ancient Rome, the husband and father having the *patria potestas* with power of life and death over wife and children. In the feudal centuries

woman was also dependent, the property of her husband, and possessed of very few rights. She had no way of earning a living, and submitted perforce to the domination of the male head of the house.

The Industrial Revolution, which separated the worker from his tools and made him a wage earner, gave him so little in real wages that his wife and children were forced into the factory to eke out the family income. A woman can throw a lever to control a cotton loom as well as a man can. With growing opportunities to earn money, women became less willing to submit to tyranny. When education became popularized in the nineteenth century, and it was discovered that women have brains, the self-respect of women was increased and they became more unwilling to submit to arbitrary domination. As more and more fields of employment were opened to them, the old ties that bound them to tyrannical husbands became weaker, and divorces began to multiply.

For a long time, in spite of growing economic independence, social and religious sanctions prevented a rapid spread of divorce. But gradually the spread from individual to individual of the ideas of militant leaders began to change the attitude of society on the subject of divorce.

The forces that hold man and wife together may be summarized as follows:

1. Economic dependence
2. Satisfaction or joy in the union
3. Social sanctions
4. Religious sanctions
5. Children

If the first has disappeared, and the second follows it, the remaining considerations will be strong or weak according to the training and outlook of the parties concerned. Lately social scientists and students of childhood have raised the question whether two people who are thoroughly incompatible, and who dislike each other, are suited to rear children, and whether it is good for a child to grow up in an atmosphere of hatred and dissension such as would undoubtedly result therefrom.

Without any desire to enter the controversial fields of this question, I propose to submit some facts and inferences as to whither we are trending in the matter of divorce. The conclusions are

based on a mathematical analysis of the facts, and do not necessarily represent my own opinions on what is desirable.

The data shown here on marriage and divorce in the United States were taken from the bulletins on that subject published by the Department of Commerce. The study was made in 1926, and the data available since then enable us to determine the accuracy of the predictions on marriage and divorce based on the trend line, to 1929. The following tables were compiled from the bulletins of the Department of Commerce:

TABLE I. MARRIAGE RATES PER 1,000 OF THE POPULATION OF THE UNITED STATES, 1887-1926

Year	No. of Marriages per 1,000	Deviation from Average	Year	No. of Marriages per 1,000	Deviation from Average
1887	8.7	0.8	1900	9.3	0.2
1888	8.8	0.7	1901	9.6	0.1
1889	9.1	0.4	1902	9.8	0.3
1890	9.0	0.5	1903	10.1	0.6
1891	9.2	0.3	1904	9.9	0.4
1892	9.1	0.4	1905	10.0	0.5
1893	9.0	0.5	1906	10.5	1.0
1894	8.6	0.9	1916	10.7	1.2
1895	8.9	0.6	1922	10.3	0.8
1896	9.0	0.5	1923	11.0	1.4
1897	8.9	0.6	1924	10.5	1.0
1898	8.8	0.7	1925	10.3	0.8
1899	9.0	0.5	1926	10.3	0.8
Average for Entire Table				9.5	0.6

Coefficient of variation $= \frac{0.6}{9.5} = 0.06$ approximately.

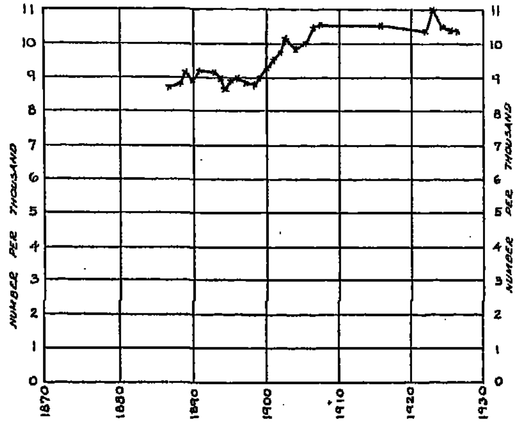
TABLE II. DIVORCE RATES PER 1,000 OF THE POPULATION OF THE UNITED STATES, 1870-1926

Year	No. of Divorces per 1,000	Deviation from Average	Year	No. of Divorces per 1,000	Deviation from Average
1870	0.28	0.76	1916	1.13	0.09
1880	0.39	0.65	1922	1.36	0.32
1890	0.53	0.51	1923	1.49	0.45
1900	0.73	0.31	1924	1.50	0.46
1906	0.84	0.20	1925	1.52	0.48
			1926	1.54	0.50
Average for Entire Table				1.04	0.43

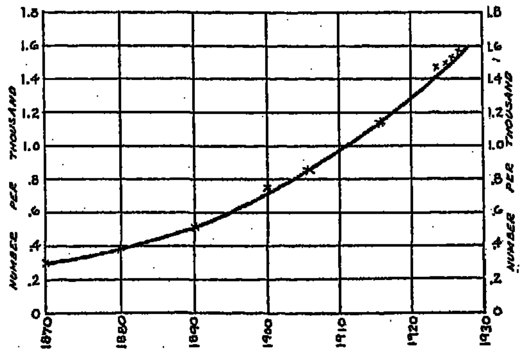
Coefficient of variation $= \frac{0.43}{1.04} = 0.41$ approximately.

Tables I and II show that the marriage rate is relatively constant for the years covered, whereas the divorce rate has varied greatly.

By plotting the data of the above tables it is easy to see that the marriage rate fluctuates slightly, mainly with the business cycle; while the divorce rate curve strongly suggests the compound interest curve. The marriage rate from 1887 to 1898 fluctuated about the rate of 9 per thousand of the population. The rapid economic expansion of the United States raised the rate rapidly from 1899 to 1906, so that it has since been fluctuating approximately about 10.5 per thousand. The graphs of Table I and Table II are herewith presented.



NUMBER OF MARRIAGES PER THOUSAND OF POPULATION—
UNITED STATES—1887-1926



NUMBER OF DIVORCES PER THOUSAND OF POPULATION—
UNITED STATES—1870-1926

It should be stated that the data given are not complete, but are all that were available through the Department of Commerce.

By drawing the divorce rate curve, using an enlarged scale and spline passing through most of the points as shown on the graph, a table of values of the rates corresponding to each five years is read from the curve. Applying finite differences to the resulting data, the ratios of the y's (rates) are approximately equal, showing that the best fitting curve is of the form

$$y = a\, b^{x}$$

Applying logarithms, we get

$$\log y = \log a + x \log b.$$

This equation is linear in x and $\log y$. So the equations of condition

$$\log b\Sigma x^2 + \log a\Sigma x = \Sigma x \log y$$
$$\log b\Sigma x + n \log a = \Sigma \log y$$

were applied to the data of Table II. The result was the equation of divorce trend

$$y = 0.2779 \times 1.03057^{x},$$

where $x = 1$ for 1870, $x = 11$ for 1880, etc.

If we assume a straight line trend for the marriage rate, and fit a straight line to the data given in Table I by the method of least squares, we arrive at the equation of trend

$$y = 0.00534x + 10.53,$$

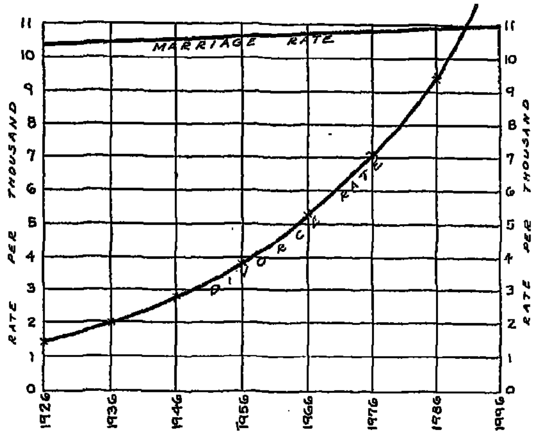
where x is the number of the year $(1906 = 0)$, and y is the corresponding marriage rate.

The present trend of marriage and divorce. If the forces governing these rates remain as fixed as they evidently have been since 1870, our two equations tell us a remarkable story of whither we are trending in this social field. Using the two equations derived,

$$y = 0.2779 \times 1.03057^{x}$$
$$\text{and } y = 0.00534x + 10.53$$

we may plot these trend lines, and hazard a prediction of the fate of marriage and divorce as social institutions. Of course, it may very well be that economic changes will arise which will bring in modifying factors of great importance. Social and religious sanctions may change in their relative importance as factors. All that we claim is that if present trends persist as indicated in the graph,

by 1990 divorce will be as common as marriage. That means, of course, that by that time society will regard the marriage contract as it regards any other contract, and not, as it is now theoretically regarded—indissoluble.



PREDICTION CURVES—MARRIAGE AND DIVORCE

The change taking place in the attitude of society toward divorce is reflected in the changing causes for which divorce is granted. In the period 1887 to 1896, about 17.5% of the decrees were granted for adultery. This cause furnished only 9.3% of the decrees in 1926. Divorces granted for cruelty constituted 19.8% of all those granted in the earlier period, and 38.5% of those granted in 1926. Evidently cruelty and incompatibility are now looked upon more generally as valid causes for divorce than they were in the earlier period.

As a test of the reliability of the equation of trend of the divorce rate, it may be interesting to compare the predicted rate for the years 1927, 1928, and 1929 with the actual rates reported for the same years. The equation of the divorce rate trend is

$$y = 0.2779 \times 1.03057^x,$$

where $x = 1$ for 1870. For 1927, 1928, and 1929 $x = 58$, 59, and 60, respectively. Substituting these values for x in the above equation, we get for y:

1.59, 1.64, and 1.69

Actual values are: 1.62, 1.63, and 1.66.

The agreement is very close indeed. The figures reported by the Department of Commerce, Bureau of the Census, vary from the predicted rates by +.03, —.01, and +.03, respectively.

In intellectual and educational fields. The main outlines of scientific knowledge were fairly well indicated by the close of the nineteenth century. The great generalizers of that time had worked out man's place in nature, as a part of evolving life on our planet. Astronomy, using the Newtonian conceptions, had achieved so many triumphs in predicting celestial events, with a minute factor of error, that its authority was practically unchallenged. Physics, chemistry, biology, geology, engineering, and the other branches of scientific knowledge are in their elements so well known to every educated man and woman that it is needless to mention specific subjects. But this realm of science is now being split up and subdivided so minutely that a scientist may spend the major portion of his life in one small subdivision of physics or chemistry, such as the problems of radio-activity. This minute specialization calls for a small army of intensely trained specialists, who become absorbed in their specialities and whose language becomes so technical that the great mass of mankind fails to get even an inkling of the meaning of their activities. The conquests of science leave the mass of the people almost unaffected, except as they enjoy the practical by-products, such as the radio, flying machines, and television. Meanwhile, primitive explanations of the origin of the earth and of man which, of course, flatly contradict the findings of careful scientific research, continue to be taught. A recent tendency in the United States is for the ignorant masses through state legislatures to interfere with the teaching of scientific truth in our public schools, and to try to set up their untutored conceptions as the standard for the young through legislative fiat. The Scopes trial in Tennessee is one of the results. One enterprising state legislator even introduced a bill to establish the value of π as 3.1, because "the children find 3.1416 an unnecessarily troublesome number to handle."

The struggle of fundamentalists to make the teaching in the public schools conform to their own interpretations instead of allowing free investigation of truth is matched by another unbalanced development of the present time. Morality and social conscience lag so far behind the achievements of science that the very existence of our civilization is threatened. Science has harnessed

such tremendous forces that our ability to create or destroy has been multiplied manyfold. With airplanes capable of carrying seventy men, explosives capable of shattering a fortress and poison gas capable of wiping out a city, the destruction that must ensue in any future war is enough to stagger the imagination of anyone capable of visualizing it even faintly. And yet, we find in the opening months of 1932 the nations more heavily armed and war preparations greater than in 1914. Greed, the mainspring of war, is as active to-day as ever.

> Careless seems the Great Observer,
> History's pages but record
> One death grapple in the darkness
> Twixt old systems and the Word;
> Truth forever on the scaffold,
> Wrong forever on the throne.
> Yet that scaffold sways the future,
> And behind the dim unknown
> Standeth God within the shadow
> Keeping watch above His own.

It may indeed be that the present dominant races are not worthy to survive; that they will persist in their struggle for exclusive domination and exploitation of subject races, and go down in the desolating wars inevitably resulting. There are available not enough data of the growth of an enlightened world view to draw any prediction curve, but if a potential destructiveness curve were drawn, combining by proper weights the factors of quantity of armament and man power with relative destructiveness, it would show, not a normal growth shape, but a compound accrual curve now mounting very rapidly. Can we as teachers do anything to save the world from this growing madness?

In view of the present state of the world and the trends and forces now operating, it seems that the major objects of education should be the following:

(1) To orient the pupil in the scientific environment of the present century, giving him some acquaintance with the major fields of human knowledge.

(2) To give him at least a rudimentary knowledge of the economic structure of modern society and of the way in which economic forces control politics and governmental policy, and engender wars.

(3) To inspire him with a desire to bring about the earthly paradise, to banish war from the earth, and to turn the conquests of science to the mutual benefit of man.

To this end it is needful that teachers and pupils alike should cultivate the habit of open-mindedness; that they should welcome new ideas and be willing to examine them, and never reject them merely because they seem to contradict old systems of thought and prejudices around which their emotions have been allowed to twine. It is also highly important that we make our thinking clear, definite, and, wherever possible, quantitative. Educational measurements are a step in this direction. Behaviorism and psychoanalysis, the study of conditioned reflexes, and all the other phases of educational theory that tend toward quantitative methods should be welcomed and investigated by all teachers who value truth and human welfare more than tradition and convention. In some communities it is hard to do this, because even now

We are traitors to our sires
Smothering in their holy ashes
Freedom's new lit altar-fires.
Shall we make their creed our jailer?
Shall we in our haste to slay
From the tombs of the old prophets
Steal the funeral lamps away
To light up the martyr faggots
Round the prophets of to-day?

Still before us glean Truth's camp fires,
We ourselves must Pilgrims be,
Launch *our* Mayflowers and steer boldly
Through the desperate wintry sea,
Nor attempt the future's portals
With the past's blood-rusted key.

ADVENTURES IN ALGEBRA, I

By HELEN M. WALKER

Teachers College, Columbia University, New York City

The fact is that science was undertaken as an intellectual adventure: it was an attempt to find out how far nature could be described in mathematical terms.—J. W. N. SULLIVAN.

I. ALGEBRAIC JUNGLES

Two points of view. If algebra is a medium in which boys and girls of average ability cannot learn to think for themselves then algebra must in time disappear from the high school curriculum.

There are two points of view. Some hold that the algebraic hinterland is a territory so mysterious, so full of pitfalls, so entangled with restricting rules and conventions, so unfamiliar and so terrifying in atmosphere, that only the hardiest of youths can find his way about in it if he is allowed to wander with any freedom. Believing that beyond this perilous jungle lie all the achievements of modern science and invention, believing that civilization must make its way through algebraic thickets in order to survive, some teachers say—at least by implication—to their classes: "We are about to enter a country rich in the value of its products to the engineer, the scientist, the statistician, the electrician, the inventor, the business executive. It is necessary that you travel through it, because only by requiring all her children to go this path can the world be sure that there will arrive on the other side of this wilderness those who are to use the mysteries of algebra for the advancement of civilization. Now this path is too tortuous for you to try to discover any part of it for yourself, and if you should attempt to go down any uncharted bypaths, if you look to right or left, you will certainly be scratched and torn. If you will surrender yourself to my guidance, will do exactly as I tell you, follow all the rules, learn all the tricks of the trade, and observe them whether they have meaning for you or not, I will conduct you safely and with a modicum of inconvenience to the gate at the other side of the forest, where an officer will examine you and issue a passport ad-

mitting you to a land where you will be free to forget the experiences of the journey. All that is required is that you shall follow to the letter the rules which will be furnished you, no matter how meaningless they may seem. But do not look behind too many bushes; do not become too much interested in the scenery; for if your curiosity should be aroused you might travel more slowly and we might fail to arrive at the gate in time to secure the all-important passport." Many guides with this philosophy have achieved great skill in conducting parties expeditiously over the set course. Their records for time consumed in the trip, distance covered, and proportion of passports issued are remarkable. For the most part we do not know how many of these wayfarers return later of their own choice to explore the surrounding country. Nor have we much information as to the character of the impressions which these travelers carry away with them. Apparently most of them are not much changed by their journey except as some exhibit an increased docility and ability to follow instructions without questioning their meaning, and except as others rebel at the hardships of the journey. The best travelers, the hardy few who can find their own way anywhere, of course find opportunities for independent exploration and secure satisfying adventures denied the larger group who keep their eyes on the prescribed path.

Observing the results of such algebraic journeyings, many who are mapping out programs for youth feel that the algebraic wilderness should be omitted from their itinerary; that it provides less opportunity for adventure, for discovery, for the joy of meeting and overcoming obstacles, less satisfaction through achievement, less understanding of life and the world, less scope for personal initiative, than other regions. If this view is correct, then even those of us who have found the most delight in algebraic wanderings will be obliged to admit the error of forcing high school children to sojourn for a year in an arid region when there are near by many more profitable and stimulating countries, for time is short and many pleasant regions beckon.

On the other hand, some teachers hold that the route traveled is of less importance than the experiences encountered on the way, and that it is possible for the child of mediocre ability to make discoveries in algebra, to learn to find his own way through the thicket, to feel the thrill of mathematical pioneering. These guides maintain that the casualties are fewer and less serious when the

journey is attempted in the spirit of discovery, and that more of the travelers return for later visits to the region. Especially do they believe that those who have felt a responsibility for mapping their own pathway come out with a better understanding of the nature of the terrain and the value of its products, with a clearer conception of its relationship to other countries, with happier memories of the experience and with new views of the rest of the vast continent of human life of which this is a portion.

If there is any soundness in this second position, then those of us who want to preserve algebra in the high school curriculum may well set ourselves the task of finding ways to help boys and girls become mathematical discoverers and pioneers.

II. Preparing for Algebraic Exploration

Unnecessary baggage. The man who is setting out on a voyage of discovery and adventure must look carefully over his equipment to eliminate all that might hamper his movements. At the outset of an algebra course many a boy or girl walks with heavy step because of a load of fear derived partly from previous mishaps in arithmetic and partly from tales which other algebra travelers have told of hardships encountered and defeats sustained. Discoveries are not made by men whose minds are occupied by fears. Nothing is much more disastrous to the solution of a mathematical problem than to approach it with a divided mind, and that pupil is traveling the road to failure whose mathematical thinking is accompanied by such an undertone as: "I'll probably get this wrong. I usually make mistakes. Perhaps I have made one already. I had better stop and go back to the beginning for fear I have made a mistake. My work never comes out right." If a pupil comes out of an algebra course with a confirmed sense of defeat and failure, or even with a growing impatience and distrust of himself, he has probably lost more than he has gained from the experience. When a pupil is truly unequal to mathematical study we should free him from the requirement, and let him attempt studies more appropriate to his abilities. When he is able enough for the work but is paralyzed by old fears and inhibitions, we should study to set him free from these. This involves studying his situation carefully, discovering the points of trouble, helping him overcome them, and then giving him opportunity for real and satisfying achievement.

The goal. The selection of a challenging but impossible goal usually results in disillusionment. The selection of an unimportant goal results in boredom. We may easily err in either direction.

Algebra is the foundation stone of the sciences, we tell our classes on the first day of the term. Without its aid our big bridges could not be built; deep sea navigation would be impossible; modern discoveries in radio, in electricity, in mechanics would not have been made; insurance companies could not operate. Quite naturally the children suppose we are promising them a glimpse into the secrets of the universe. The facts are true but the promise is specious. When a little girl I went on a visit to Colorado, and a pedagogically minded neighbor attempted to instruct me in advance as to how a mountain would look. She so impressed me with the enormous height of the Rockies that I expected to see them spring abruptly from the ground at my feet, rising with perpendicular cliffs to heights which the eye could not follow. The result was that in reality the Rockies seemed quite negligible and disappointing. Just so, by promising too much, do teachers sometimes take the fine edge from the thrill of intellectual discovery which is the right of the student of algebra. No informed and intelligent person would to-day discount the importance of the applications of algebra in the adult world, but to fall back upon that indubitable fact as the chief source of motivation for the teaching of algebra in the high school would be unresourceful; and to expect boys and girls to find algebraic manipulations entrancing, merely because electrical engineers need to know algebra, would be naïve.

On the other hand, to tell a class at the beginning of the term that they are expected to become discoverers of mathematical laws would probably be disastrous. In fact, the class is likely at the outset to be so much preoccupied with the new experiences they are having that a discussion of the reasons for studying algebra would not interest them much. But the teacher must know for what he is working, whether his goal is manipulative skill or ability to think in algebraic language; and in the course of time he must convince his class that the goal is worthy and is attainable.

Choice of route. An emphasis upon discovery as a goal for the teaching of algebra has little to do with the general choice of subject matter or the major order of topics. Almost any material may be made to yield opportunities for mathematical adventure if the teacher plans for it. He must of course set the stage. He must

know when to withhold relevant information; when to feign ignorance or forgetfulness of some important mathematical fact. He must plan the questions by which he leads his pupils to the brink of a new mathematical idea as carefully as a guide would choose the trail through a forest so as to bring a party out at the right spot to catch the finest views, and must then, like the guide, step back at the psychological moment so that his charges can make the final discovery for themselves. He must provide the most advantageous conditions for thinking and yet leave the child free to do his own thinking in his own way. Children may be taken to the Rockies but they should have the satisfaction of believing that the mountains are their own peculiar discovery.

General procedure. In organizing an algebra course to emphasize the element of discovery, it is of first importance that *pupils shall learn to use symbolism as a language in which they may express their own ideas.* Children who can use symbolism in this creative fashion are not likely to think of algebra as a set of meaningless and arbitrary rules. Moreover they will have taken an important step toward the appreciation of scientific method. Most high school students recognize the component parts of a formula about as a beginning student of French or German recognizes word units, without deriving any meaning from the totality. Or they may progress to the point where they can change the subject of a formula, solve an equation, or manipulate fractions; in other words they can read statements which are presented to them. Free composition in the language of algebra is a further stage too often overlooked in our teaching and not to be attained without careful preparation. To develop power in this creative use of symbolism, several things must be kept in mind.

1. If a child is to be successful in any sort of composition, English, French, German, musical, or algebraic, *he must have ideas appropriate for expression in that medium, and they must be ideas which have significance for him.* The commendable desire of teachers and textbook writers to make algebra practical often results in the use of formulas which are of real importance in science or in business but which have little meaning to most children. Such illustrations may impress a supervisory officer or a textbook committee, but do not facilitate vigorous thinking on the part of pupils. The relations underlying the formulas most used in physics, biometry, astronomy, insurance, or business are not as a rule sufficiently

real to the imagination of the ninth grader that he can *compose* algebraic formulas in these fields. There are, however, a large number of situations involving quantitative relations which are so real to the ordinary child that they can be used as opportunities for him to express himself algebraically. Research directed toward finding such situations real to the imagination of children in their early teens, and toward creating a body of material which can be used for practice in writing formulas would no doubt be at least as valuable for teaching purposes as research into the uses of algebra by adults.

2. *The idea must always precede its symbolic statement.* This is of the utmost importance. Simple as it sounds, if teachers should accept this as a guiding principle it would require them to change almost all their practices in the teaching of algebra. Textbooks written on this principle would be quite different from many of those now available. This would usually demand a leisurely period of exploration while a new concept is being developed, and during this exploratory period clarity of meaning would be sought in every possible way, with an emphasis upon forms different from the symbolic statements to be finally attained. For example, in teaching logarithms, all the operations might be first performed on a table of powers of two, so that the general processes are clear to the child before he is given the logarithmic notation. In this period the pupil is given problems so arranged that he can work out their solution and so chosen that they will lead naturally to the generalization desired. *He is taught to look for this generalization,* to be on the alert to see a general plan beyond the specific problems he is solving; he is encouraged to make his own rules. It goes without saying that such pupil-made rules or generalizations are accepted with all due respect however crudely stated, and the child is vouchsafed a sense of achievement because of his attempt. Undue emphasis on form at this early stage will completely cut off pupil-initiative, but after the pupils have made their somewhat awkward formulation of principle they must be helped to get a clear and simple statement; after they have mastered an idea they must be helped to express it in good symbolic form, and must learn to derive satisfaction from this statement. The arithmetic processes, addition of simple fractions, and the like, offer excellent opportunity for such generalizations, thus furnishing a review of arithmetic while attention is directed toward the making of algebraic formulas, and the use of

algebraic language in a situation real and understandable to the child.

When a pupil has had a hand in formulating such principles they have meaning for him and he does not forget them readily. The essential thing is that he shall form the habit of looking for principles, shall feel the responsibility of finding his own way. In classes where this spirit has prevailed, I have heard children of very mediocre natural ability say, "That sounds like a hard problem. I think I shall have to make a formula for it so I can think better about it."

At the beginning of any new topic, this method seems time-consuming and grossly inefficient. Progress is slow and the prospect of passing any set examinations seems uncertain. But one of its advantages is that the ideas which seem to the children as their own have more significance for them and are forgotten less easily. Less drill and less review are required to hold the same degree of skill. Moreover when children do forget a fact or a technique, they have command of resources for reviewing it; they have been over the path before with their eyes open and they can retrace it.

Experiment has convinced the writer that this kind of work, slow as it seems in the beginning, will allow a class of average ability to accomplish more in the course of a year, to cover more topics with greater thoroughness, than can be done under the most brilliant teaching which emphasizes drill and routine and acceptance of rules, and which does not challenge the child to formulate principles for himself.

The temper of the guide. The success of such exploration lies largely in the hands of the guide. He must be one who takes delight in a new idea himself, and who considers vigorous thinking a challenging adventure. He must have faith that *children like to think* when their thinking is treated with respect and when it has a fair prospect of success. He must feel that finding and stating an important generalization is a worthy achievement, even if that generalization is one long familiar to him. He must be willing to wait, to let a pupil express crudely what the teacher could express with finesse, to subordinate polished work to vigorous thinking. He must be resourceful in seeing worth in statements far different from those he was expecting, must be flexible in his ability to utilize for teaching purposes every opening which his class offers. He must feel that what happens to the pupil himself is of paramount im-

portance. Above all, he must be able to put himself *en rapport* with his pupils and to win their coöperation, so that they will be willing to venture, secure in the knowledge that even if they are grossly wrong, their attempt will be treated with courtesy.

Algebra can be taught in such a way that boys and girls may have the thrill of mathematical discovery, the joy of intellectual pioneering. The importance of helping pupils to discover geometry proofs is generally recognized. Unfortunately, too many supervisory officers still think that any good drillmaster can teach algebra, assuming that skills and techniques are the primary goals of instruction in this subject. Any good drillmaster can teach most pupils to go through a routine of manipulations, to obey the rules, to get the answers; but in a year or so the rules are forgotten, the routine vanished, and only the memory of a trying experience remains. The spirit of every mathematics classroom should be that *of adventure and exploration. If algebra is to have life values we* must find a way to give the child a more creative rôle.

ADVENTURES IN ALGEBRA, II

INSTANCES OF CREATIVE THINKING IN ALGEBRA

By VERA SANFORD

School of Education, Western Reserve University, Cleveland, Ohio

We hear a great deal, nowadays, about creative work in literature and in the other arts, but the phrase "creative mathematics" is used only rarely. Skeptics will say that nothing in creative mathematics can be done by students of high school age. They will admit that Pascal wrote his essay on conics at the age of sixteen, but they lose no time in noting that the field in which Pascal worked has been thoroughly surveyed, and that new worlds to conquer lie at a great distance. If they are teachers of mathematics, they may quote their difficulties with the perennial angle trisectors, and they then ask if "creative work" may not merely set more students in pursuit of a will-o'-the-wisp. Indeed, to many people outside the field, and perhaps to a few within it, mathematics is a subject in which one follows the reasoning of others and practices techniques which they have devised, but in which one does little actual thinking beyond the solution of verbal problems in algebra or the proving of original exercises in demonstrative geometry. To many people, the subject consists of what Whitehead calls "inert ideas," that is, "ideas received into the mind without being utilized, or tested, or thrown into fresh combinations."[1] As teachers of mathematics, we are aware that this is not the case. Discovery implies that the thing is new to the individual—not necessarily that it is new to the race. Critical thinking need not be restricted to problem solving *per se.* To quote Professor C. B. Upton in the report of the National Committee on Mathematical Requirements, ". . . a great deal of this problem solving attitude can come from the right method of handling and thinking about the operations themselves."[2] Concepts which are treated in this way will not be *inert.*

[1] A. N. Whitehead, *The Aims of Education and Other Essays,* p. 2. New York, 1929.
[2] *Reorganization of Mathematics in Secondary Education,* p. 365. 1923.

It is the purpose of this chapter to supplement Professor Walker's "Adventures in Algebra, I" by outlining and discussing certain actual classroom episodes in which the subject matter was in the main concerned with the ordinary techniques and concepts of elementary algebra, and in which the work was in some cases undertaken on the initiative of the students themselves.

Such adventures vary with the mentality and training of the student. A rare achievement for one is a commonplace for another. On the other hand, these incidents have like elements in their initiation and outcome. They begin as a self-assumed task, carried out with an uncertainty as to the result, and they terminate with the sensation that this particular accomplishment is the student's own by right of his discovery. The result is often a mosaic made by the coöperative effort of the whole group. An hypothesis, crude in its first phrasing, is polished and refined until a creditable product results. During the process, all members of the class are at work —some making suggestions, seeing farther than most; others acting as critics or as judges. Each has the privilege and the obligation of expressing his own opinion as the work progresses, and all are responsible for the outcome. In point of time consumed, Professor Walker is wise in her caution that progress is slow at first, but it gathers speed, and, at the end, the students seem to be at once more secure in their knowledge and more resourceful in their attack on other problem situations. The adventures of one year may be duplicated and bettered in the next, for a teacher can skillfully set the stage for a revival of the old play with the new cast, and each repetition yearly offers new and varied interpretations of the old idea.

The need of symbols and of names. A seventh grade class had discovered the relation of the angles of a triangle by a combination of intuition and experiment. The principle was expressed in words, and then, letting A, B, and C represent the sizes of the three angles in degrees, the law was written in algebra as

$$A + B + C = 180^\circ.$$

The teacher then asked whether the statement could be written in different form, and a pupil phrased it as

$$A + B = 180^\circ - C.$$

The other permutations of this scheme followed readily. Then a

student, who was a bit more thoughtful than the rest of the class, wrote

$$A = 180^\circ - B \text{ and } C.$$

When questioned, he said that what he meant was to add B and C and then to subtract their sum from 180, but if he had said $A = 180^\circ - B + C$, then this would mean to subtract B from 180° and then add C. Neither he nor the others were satisfied with this mixture of algebra and English, and the teacher introduced, in a simple and natural way, the idea of parentheses. Here a piece of invention was found less worthy than the to-then-unknown symbolism. The suggestion of the *and* was forgotten and the reason for parentheses appealed to the class as valid and worth while.

At a later date, the same students were solving equations in the form $4x = 12$. The method decided upon after some thought and experiment was to divide both members of the equation by 4. When equations of the type $\frac{1}{2}x = 12$ were introduced, both members were multiplied by 2, but one boy maintained that he could solve both kinds by one rule,—"Divide both sides by the number of x." He explained that to divide by $\frac{1}{2}$ was equivalent to multiplying by 2. The class accepted the statement but objected to the expression "the number of x," and the word coefficient was introduced as the term adopted for this expression by mathematicians.

The solution of simple equations. Discovery in algebra may sometimes take the form of summarizing many separate problems in a general rule as was done in a somewhat casual way in the last instance cited. At times, however, this work forms the basis for class work over a considerable period. In general, a statement framed by the class or by a single member of the class carries more conviction than one taken from a textbook or dictated by the teacher. A ninth grade class had been solving first degree equations in one unknown by reasoning each as a distinct problem. If $x + 3 = 9$, then since three more than x is nine, x itself must be three less than nine or six. If $4y = 12$, then y is one-fourth of twelve, or three. No mention had been made of laws or axioms. It was the teacher's intention to bring the class to an independent formulation of the method of work by developing a rule for each of the four cases. One student (I.Q. 93), however, anticipated the rule. On the second day of the work, she showed an ease and

speed in solving equations that made her classmates curious to know how she did it. She explained "I do just the opposite of what the equation says. If it says x plus 4, I subtract 4. If it says 4 times x, I divide by 4." Such a formulation does not come by accident. It is the result of a process akin to the one by which the scientist surveys his data and uses them as a basis for making an hypothesis. The next step in the treatment was to try the rule on equations of all different types, and the class reached the conclusion that the rule was correct because it was successful in all instances.

In another class, where the teacher had built up a similar situation, given many different cases from which to derive the rule, a boy said, "The x in this problem is disguised. I can find its value only by undoing the disguise. If it is disguised by addition, you untie it by subtraction, and so on."

In both cases, we have the generalization from many specific cases into a rule that must be tested for its validity but which, if valid, may be adopted as an aid in computation. Skill coupled with understanding is better than skill alone. The process of deriving a rule and phrasing it for oneself, instead of passively accepting it, has much to commend it.

Excursions beyond the usual confines of the subject matter. A group of eighth grade students were studying directed numbers at the close of their year's work. The choice of the subject was governed by considerations that do not concern us here. The writer makes no brief for the use of this topic in this grade. The class began with the concept that directed numbers are used when you wish to express quantities that are the opposites of each other. Various instances were cited to show that the idea was a familiar one. The usual examples were given, with the addition of one picked up on a Saturday visit to the oculist, that + and − signs were used in writing prescriptions according to whether eyes are nearsighted or farsighted. Then a student volunteered the information that there was something that was positive when a street car started and negative when it stopped. It was not the time, not the distance, not the speed. The time was passing at a constant rate. The distance increased to the end of the run. The speed began and ended at zero, but was positive in between unless the car should go backward. Finally another boy caught the idea and suggested that the first boy meant the "acceleration and deceleration of

the car." By that time the class was keenly interested. A graph was sketched to show the relation of the velocity to the time throughout the run. Then, on this basis, the acceleration curve was roughly plotted by the sponsor of the first suggestion. The demand for other instances of this type gave rise to the brief study of formulas familiar through work in general science, $s = \frac{1}{2} gt^2$, $v = gt$, and the unfamiliar one, $a = g$. Activities in algebra sometimes impinge on territory which we are not at present competent really to explore, but they have worth-while results in removing the unfortunate impression that "the book" holds all there is to know about the subject.

Refining a rule. The multiplication of directed numbers was a subject partially remembered by a student who was repeating the course. He announced, "Like signs give plus answers and unlike signs give minus answers." Here the teacher had the choice of amending the rule or of bringing the class to do so, and, believing the latter to be the more effective course, she provided several problems in which the rule could be used with success. These ended with the problem $(-1)\ (-2)\ (-3)\ (-4)$ which the students said had the value $+24$. The (-1) was then erased, and those who were blindly using the rule kept to the same result. But one pupil who had multiplied step by step, claimed that the product was -24. Here was a dilemma—the result of the one-step-at-a-time method versus the like-signs rule. The rule was rephrased: "If you have an even number of factors, like signs give products that are positive and unlike signs give products that are negative." This stood only for a moment for a student proposed the problem $(+2)\ (+3)\ (+4)$ where the product was evidently $+24$, and the teacher contributed $(+2)\ (-3)\ (-4)$. Solved by the one-step-at-a-time method, the results again failed to fit the rule. After refining the process by gradual stages, it was agreed that if the number of negative factors was odd the product would be negative, but in all other cases the product would be positive.

It cannot be claimed that the students' mastery of the problem was perfect, but it is probable that there was less partial memory of the rule than would have been the case had the teacher stated it *ex cathedra* and then spent the rest of the hour in the application of the rule. Fewer problems were solved, it is true, but the attention of the pupils was well concentrated on the essential element in the case.

Definitions are arbitrary. A class in intermediate algebra had met in the room the preceding period, and there remained on the board the statement $x^0 = 1$. The ninth grade urged its investigation. They suggested that it meant x degrees is zero, but they suspected that it had a deeper meaning. The teacher had three possible courses of action: to tell them that they would learn about it later, to explain it briefly, or to bring them to an understanding of this definition of a zero exponent. The first way has little to commend it. The second might have been dictated by an exacting course of study, but in this case time permitted and the teacher, firmly believing in the *carpe diem* philosophy, felt that ten or fifteen minutes devoted to the subject would pay dividends both then and later. Accordingly, she reviewed the meaning of x^4, x^3, etc. The class had never met the exponent rule for division, but the students derived it on the spot from the consideration of simple examples such as $\frac{x^4}{x}, \frac{x^4}{x^2}, \frac{x^4}{x^3}$. The rule was applied to several other cases, and the correct result was obtained each time. The teacher then put the question, "What is the value of $\frac{x^2}{x^2}$?" By the rule, the result was x^0. "But," said the teacher, "what is the value of $x^2 \div x^2$?" The answer was "One" (1). The results then stood as is shown in the following:

$$\frac{x^2}{x^2} = x^0 \qquad\qquad x^2 \div x^2 = 1.$$

A student called attention to the fact that these were different ways of stating the same problem, so the answers ought to be alike. A boy said, "We know that $x^2 \div x^2 = 1$ is right." After a little thought, the suggestion was made that if x^0 was correct also, then its value must be 1. Why? Simply because this was necessary in order to make the subtraction rule for exponents hold when the exponents were identical. Accordingly, it was agreed that they would say that x^0 is 1, and that they would accept this as a definition that was a consequence of the rule. Only a few students would keep this as a part of their permanent equipment—much additional practice would be needed—but many of the class would remember that certain things are defined in particular ways because it is expedient to do so, and that definitions on the whole are arbitrary things.

The power of symbols. At a later part of the course, another ninth grade was definitely working with the division of numbers in the form a^n. Their home work consisted of a large number of problems of the type $\frac{y^5}{y^3}$. In a few cases, however, the exponent in the denominator was larger than that of the same letter in the numerator. On coming into class, Sophie asked if she might say that x^{-2} stands for $\frac{1}{x^2}$. Now, definitions are arbitrary, but they call for acceptance on the part of the group before they can be used. Sophie explained that if this definition could be adopted, she could then do all the problems in the assignment by the one rule of subtracting the exponent of the letter in the denominator from that of the same letter in the numerator, but if the notation could not be used, she would have to stop each time to consider whether or not the exponent in the numerator was greater than that in the denominator. The first procedure would be less time-consuming and if the teacher would accept it, she would prefer it. The problem was set before the class, who asked the meaning of a^{-1}, a^{-3}, a^{-4}, etc., and who unanimously decided to accept Sophie's exponents. The teacher suspected that they had a deep-seated aversion to fractions. "Sophie's exponents" were tested in various ways. For instance, did the multiplication rule hold? Did $x^{-3} \cdot x^{-2} = x^{-5}$? Surely, because it was simply another way to say $\frac{1}{x^3} \cdot \frac{1}{x^2} = \frac{1}{x^5}$. Was $x^{-3} \cdot x^4 = x$? Yes, because $\frac{1}{x^3} \cdot x^4 = x$. For several days, operations with the textbook exponents were paralleled by operations with "Sophie's exponents." The zero exponent was discovered and defined. After about a week of work in the regular course of study, Sam raised a question regarding the meaning of $x^{\frac{1}{2}}$. When asked how he thought of this, he said that he knew about numbers that were positive, negative, whole numbers, and fractions. He knew about exponents made of each of these except the fractions and he wondered what a fraction as an exponent would mean. In other words, he had seen the idea of an exponent generalized in order to make a rule apply to all possible cases. He now wanted to investigate the consequences of enlarging his symbolism still further. Few more interesting illustrations of the growth of symbolism and the stimulating effects of a good symbolism are to be found than appear in the case of exponents. To

study Sam's question, the teacher inquired about fractions. They express division. Division is the opposite of multiplication. What happens when we multiply exponents? The value of $x^4 \cdot x^4$ may be expressed as x^{4+4} or as $x^{4 \cdot 2}$. In both cases, the value is x^8. We raise x^4 to the second power by multiplying the exponent by 2. Since division is the opposite of multiplication, it would be reasonable to suppose that dividing an exponent would have the opposite effect to multiplying it. But exponents are multiplied by 2 when the numbers are to be squared. Accordingly, it is reasonable to say that when an exponent is divided by 2, we find the opposite of the square—that is, the square root—of the number. Dividing by 3 would yield the cube root, etc. The question of fractional exponents was dropped at this point. They existed. Their meaning was a matter of definition. It is interesting to note that although the class had forgotten the details of their work with negative and fractional exponents when they reached it in intermediate algebra, they greeted the subject as a thing they had once investigated briefly and they found it simple and easy to comprehend.

The study of a strange phenomenon. Students sometimes propose investigations that lead into topics not included in the syllabus. The extent to which these should be investigated depends on several factors, among them being the intrinsic possibilities of the topic suggested—will its study illustrate the use of familiar tools in investigating strange phenomena?

A class was applying its knowledge of exponents to finding the areas of various squares. Among the problems assigned for home work was finding the value of A in the formula $A = s^2$ when s is $\frac{1}{2}$. One pupil was much disturbed at the result. She knew from a diagram that her answer $\frac{1}{4}$ was correct, but she said she had never before multiplied a number by itself and found a result smaller than the original number. She was so disturbed over it that she asked the class what they thought of it. One boy answered that it was the fraction that made the trouble, and suggested trying $\frac{1}{8}$, $\frac{1}{4}$, $1\frac{1}{2}$, $1\frac{1}{4}$, etc. It then developed that the squares of some of these were less than the numbers themselves and the squares of others were greater. It certainly was not the presence of a fraction alone that accounted for this. It was found that $1^2 = 1$. Numbers and their squares were listed. It was suggested that a graph might help. From a comparison of the graph showing the squares of various numbers (i.e., $y = x^2$), with the graph showing

$y = x$, it developed that for numbers whose value is less than 1, the squares are less than the numbers themselves; for a number equal to 1, the square is equal to the number; for numbers greater than unity, the squares are greater than the numbers themselves. In itself, the problem answered a transitory question, but it fostered an attitude of inquiry on the part of the class and it confirmed the statement of the textbook that a graph helps to organize data effectively and helps in the study of the laws that lie behind given data.

The answer to a query. Two points determine a straight line. Three non-collinear points determine three straight lines. How many straight lines are determined by n points, no three being in the same straight line? This problem was proposed in the geometry class and a challenge was posted on the bulletin board. Competition was open to anyone interested. Hilda's solution (Hilda being in the ninth grade) was: "I can draw n lines around the edge. Then for each point, I can draw $n - 3$ diagonals. There are n points so I can draw $n(n - 3)$ diagonals; but each of them is counted twice, so I have $\frac{1}{2}n(n - 3)$ separate lines. This makes $n + \frac{1}{2}n(n - 3)$ lines or $\frac{2n + n^2 - 3n}{2}$ or $\frac{n^2 - n}{2}$ or $\frac{n(n-1)}{2}$ lines." This certainly illustrated Professor Walker's term "free composition in symbolism."

Ella, being in the tenth grade geometry class, set herself the task of graphing the size of the interior angle of a regular polygon. In presenting the graph to the class, she claimed that she had investigated large values of the number of sides, hoping to find an angle greater than 180°, but that thus far she had been unsuccesful. Her graph was criticized in that the continuous line had no meaning for values of n that were not integers. Then, granting that the graph had meaning only for integral values of n greater than 2, how could she expect that an interior angle of a regular polygon could be greater than 180°? If it were, the polygon would be concave all the way round. But, more cogently, since the formula is $\frac{(n - 2)180}{n}$, or $\left(\frac{1 - 2}{n}\right)$ 180, it is evident that no matter how large n becomes, something is always subtracted from the 1, hence the angle approaches but never equals 180°.

It must not be inferred that the drill aspects of algebra are to be neglected because of the emphasis on the thrill of discovery and of

creative thinking. The emphasis given the creative thinking depends upon one's purpose in teaching mathematics. But one thing is certain: When properly presented, these fundamental ideas of critical scrutiny of statements, the testing of rules, the arbitrary nature of definitions, the value of symbolism, the generalization of many cases into a few rules—these remain with the pupil long after the technical skills have vanished through lack of use.

PRESENT OPPORTUNITIES IN JUNIOR HIGH SCHOOL ALGEBRA

By HARRY C. BARBER
Phillips Exeter Academy, Exeter, New Hampshire

Making America algebra-conscious. We are all aware that our civilization is built upon science and that science talks and thinks in terms of mathematics. Yet there are many people, old and young, who recall their school algebra with no realization of the fact that it opens one of the doors to the understanding of our age. Even when they speak favorably of it they are more likely to tell about the hard and exacting tasks it affords than about the insight it gives into the scientific attitude of mind, or about the new language which opens the door to other fields and other methods of thought.

This is probably the fault of those of us who teach algebra in the secondary school, and this is the reason we are redefining algebra, changing our approaches to it, and reconsidering our purposes.

By redefining school algebra I mean redrawing its limits so as to exclude so far as possible inert ideas and to include and emphasize those ideas which are most potent in themselves and which are most effective in stimulating the minds of the students. That redefinition is going on is evident to all who read the textbooks of the past fifteen years.

Such redefinition should enable us to do what I have called making America "algebra-conscious," making that very large part of our population which goes through the junior high school grades realize that algebra is an important language which simplifies and clarifies our thinking about the things we count and measure; that this language, this set of symbols, is easy to use and that it accomplishes interesting and worth-while things from the start; that it opens the door to the world of science and its methods; and that it is a profitable thing for anyone to know.

The redefinition of algebra. This redefinition is very interesting to watch and is, I think, very promising. It makes headway

slowly against the inertia of textbooks and the habits of teachers; but it does make headway. It scrutinizes, for instance, a chapter on factoring and finds that when this is presented as a separate chapter early in the text it fails to be convincing. Such a chapter does little to show what algebra is for or how its symbols are used as aids to thought. It can at this stage be little but the manipulation of empty symbols, practice in puzzle-solving. It will never make America algebra-conscious in any good sense nor will it make the public, in our age, pro-algebra. Consequently, factoring tends to come later when and as it is actually needed.

On the other hand, the solution of simple verbal problems in which one can see that solving the equation is reasoning clearly and systematically about the numbers of the problem, by which one gets insight into his habits of thought and into scientific methods of procedure, is a process which shows what algebra can do. Consequently, it tends to fill a larger and larger place in first-year algebra. The redefined algebra centers, in fact, around the equation and its uses.

These, then—the equation, the formula, and the algebraic analysis of simple problems—are the parts of algebra which are to be moved back from the ninth school year to the seventh and eighth, where, it is hoped, they are to claim throughout much of the time the occasional attention of the student. They should appear, of course, when and where they can prove their usefulness. The kinds of equations which should be used in grades seven and eight, are indicated in my monograph on teaching junior high school mathematics.[1]

These too are the parts of algebra—and this point needs to be insisted upon—which in the high school should constitute a part of the education of the slower and the non-college-preparatory students. There is a strong movement among educators to require no mathematics after the eighth year. The National Committee on Mathematical Requirements is right in saying that mathematics should be required through the ninth year. The way in which we can second the effort of the Committee to bring this about, is to *create* a ninth-year course which will attract and hold the young people and which will convince the public and the educators that it is worth while. The old algebra is doomed. So long as educators think that algebra means the so-called four

[1] Barber, H. C., *Teaching Junior High School Mathematics*, Houghton, Mifflin Co., 1924.

fundamental processes, parentheses, complex fractions, radicals, factoring, and empty symbols, they will claim, justly enough, that other subjects of study are more valuable for general-course, commercial, industrial, and even college-preparatory pupils. But if our redefinition succeeds, if we can bring up a generation of people who find in algebra an indispensable symbolism for thought, an essential tool in a scientific age, simple enough in its elements to be used, understood, and enjoyed by all normal high school students, we can keep open for this generation, or at least we can open for the next one, this door to the understanding of the most significant characteristic of our age.

The organizing idea for the old school algebra was the very important one of extending our number system to include negative numbers and making clear the consequences of so doing. This plan of organization has brought algebra into disfavor. It is not an interesting point of view for fourteen-year-old minds; and furthermore it has been so concealed behind details, manipulations, and rules for signs as to be almost unrecognizable. We could somewhat remedy the situation by making a bolder attack on the main idea of extending our number system, talking more about it, bringing it into closer contact with the mind of the student. But I am sure that it is better to change the organizing idea entirely and to use instead the broader concept—the use of symbols as an aid to thought. We solve verbal problems and we see that algebraic symbols give us convenient handles for things; that solving the equations is in fact reasoning, by a sort of code, about the numbers with which we deal. This idea *is* adapted to the fourteen-year-old mind. It can make a real appeal and become quite convincing. It opens the door to fields that are immediately interesting. It is not mystifying, confusing, or unreal as are the mechanical rules for signs of the old régime. It is, in my opinion, the idea we are turning to, must turn to, in order to cure the ills we all recognize in school algebra.

A different approach. Different civilizations have used different ideas as means of educating their young. Some ideas have proved potent for this purpose, others inert. Algebra, if correctly defined, should surely prove educationally effective in a scientific age. We know in fact that it can. But even important ideas may be approached in school in ways which are not mentally stimulating. It seems to me that every approach, every introductory lesson,

ought to be planned with one end in view: that of stimulating the mind of the student, of leading him to discover, of teaching him to study, of arousing his desire to understand, of getting him to think.

This is not an easy thing to do. Thomas Edison said to Lewis Perry, Principal of Phillips Exeter Academy, that after examining ten thousand college graduates he had come to the conclusion that "thinking is unnatural." The pupil often avoids everything that requires him to think; text and teacher find it easiest to follow the same line of least resistance. Thus we miss the point of the whole thing. We content ourselves with showing the pupil how and giving him many examples to do, with the hope of fixing a habit. Such a procedure may possibly be all right in the arithmetic of the lower grades, though I do not think it is; but in the secondary grades it is so wrong that I venture to call it a parody on education. We ought instead to be devoting our whole energy to arousing interest, stimulating curiosity, compelling hard thought.

This, of course, suggests not only a different approach but also two contrasting concepts of drill; the drill of repetition and the drill of reunderstanding. These we shall consider a little later. We are concerned now with finding approaches which are mentally stimulating, which afford attractive glimpses of fields beyond, and which make each topic appear useful and interesting from the outset.

Suppose then we study these approaches and ask ourselves where each important part of first-year algebra can be introduced so as to be useful and interesting from the start. Under the new organization, we are to study the use of symbols as aids to thinking. This rules out, for the most part, manipulation of the symbols for the sake of the manipulation itself. It builds the year's work around the equation (including of course the formula) and its uses.

Our question then is: In the progressive study and use of equations where do we first meet the need for this topic and for that?

Parentheses come in naturally when we use formulas for the perimeter of a rectangle, the area of a trapezoid, the conversion of temperature from one scale to another, or when we want to think of 3 times the binomial $20 - x$. When the pupil asks what parentheses are for he should receive an immediate answer to his question. Let us hope that he has never heard the expression, "change the signs," and that he completely fails to get the stupid idea that

many children get, that a parenthesis has something to do with "changing signs." The idea of signed numbers, of extending our number system, is so important, so capable of stimulative use as a means of education, that to approach it too early by rules for signs and by such phrases as "changing signs" is an inexcusable misuse of educationally valuable material. The fact is that the student can handle many simple equations and get a good idea of what purpose algebra serves, without giving any meaning to the minus sign except subtraction and, gradually, the idea of a shortage —the idea from which algebra took its name.

When do we need vertical addition? Not in solving linear equations in one unknown. All we need for this purpose is the uniting of similar terms; an operation which is performed readily enough by the student, without any rules for signs.

Of course, he needs eventually to be led slowly into an understanding of the subtraction of a negative number. But he does not need vertical addition and subtraction until he solves linear equations in two unknowns by the method of combination. When he has reached this point, he is so familiar with the fact that the minus sign means not only subtraction as it did in arithmetic, but also shortage or opposite direction, that he can readily enough understand and accept for use the one rule he needs, namely, that for subtraction. It is my opinion, then, that vertical addition and subtraction with the rule for subtraction, not only can be but ought to be postponed until this point is reached; the point at which they are obviously useful.

And how about factoring? The removal of a monomial factor becomes useful as soon as one contemplates the formula for the perimeter of a parallelogram. It is very simple and need not even be called factoring. One merely divides as he needs to in order to rewrite $2a + 2b$ in the form $2(a + b)$. Multiplication of binomials and the inverse, division or factoring of quadratic trinomials, become useful when one studies quadratic equations and is ready to solve them. The association of these topics is natural; its advantages are obvious. It is the kind of thing we must do if we are to make each topic useful from the outset in the lives of the pupils.

Exponents beyond squares and cubes are not useful, so far as I can see, in a first course in algebra. Their theoretical discussion and their extension to include negative and fractional exponents

are surely ill adapted to fourteen-year-old minds and belong in the second course in algebra.

The meaning of the radical sign needs to be understood in connection with certain formulas which naturally come into the course. Such little manipulation and "simplification" of radicals as is necessary in these days of decimals and of square root tables, can well be postponed until the second course. Whenever these operations do come in, it seems good economy and good psychology to study them where they are obviously useful; for example, in connection with the isosceles right triangle and the 30°, 60° right triangle.

These and other similar changes in approach necessitate a considerable modification of the order of topics in the first course in algebra. Such a modification seems to me essential to our aim of making algebra serve its educational purpose in our age, of making it mentally stimulating and educationally defensible, of making America algebra-conscious.

It is particularly desirable that the new definition shall become effective before the algebra of the two lower junior high school grades has crystallized. To teach in these grades the so-called four fundamental processes, rules for signs, transposition, factoring, nests of parentheses, and the like, seems to me completely subversive of the principles we are here defending. The result in the mind of the pupil is, pretty generally, a jumble of rules and half understood processes, nearly meaningless to him and uneducative, precisely the wrong foundation upon which to build later a well-reasoned and mentally stimulating study of algebra and its uses.

If the extension of the number system was a great achievement of the race, then the approach to it ought to be a particularly invigorating experience for the young mind. It has not always proved so. On the contrary, by approaching it too mechanically, by introducing rules for signs too early, by requiring too much manipulation of empty symbols at the outset, we have, often enough, caused mental indigestion, made algebra seem futile, and aroused toward it a feeling of hostility.

So it would seem that one of our most pressing problems is that of finding a thought-provoking approach to negative numbers, an approach which reasons its way gradually as the race did in discovering the idea. When it was found that what we now call a negative number was not an imaginary or impossible sort of thing,

but that we could subtract 8 from 5 and get a "shortage" of 3, and that we could deal with this "short three" or -3 very much as we deal with ordinary "positive" numbers, then equations containing negatives began to be usable, subtraction became a general operation, and we were on our way to the complete generalization of the number system. The word "algebra" is tied up with this discovery or invention. It means "the restoration," that is, the addition of enough to make up the shortage in such an equation as $x - 8 = 10$. By means of the words shortage and surplus it is possible completely to explain the laws of signs—a thing very difficult to do in any other way.

The approach and the explanations which I have in mind are well epitomized in the following paragraphs from the *Massachusetts State Course of Study for Junior High Schools.*

> *Negative Numbers and the Four Processes.* Postpone these topics until they are actually needed. They are more difficult for the pupils than we have realized and their best development requires a considerable period of time.
>
> The simple additions and subtractions used in solving the first equations should be made without rules for signs. Later the processes should be explained step by step as illustrated below, using only the first two meanings of the minus sign, namely, subtraction and shortage. Postpone the third meaning, opposite direction (which of course includes the second), until it is needed in graphical illustration.
>
> In studying the form $15 - 3(x + 2)$ we observe that the terms in the parenthesis must be (*a*) multiplied by 3 and (*b*) the results subtracted from 15. After (*a*) has been done, the form becomes $15 - (3x + 6)$. To subtract $3x + 6$ from 15, first subtract $3x$. Have we subtracted enough? Then let us subtract the rest, the 6; then we get $15 - 3x - 6$.
>
> In the form $15 - 3(x - 4)$, step (*a*) gives us $15 - (3x - 12)$. Step (*b*); to subtract $3x - 12$ from 15, first subtract $3x$. Now we have subtracted too much, for we were to subtract 12 *less* than $3x$. We have subtracted the whole of $3x$. We have subtracted 12 too much. We correct this by adding 12. Step (*b*) now gives $15 - 3x + 12$. Repeat until understood.
>
> The pupil is now ready to understand the rules for signs in subtraction and multiplication developed as above. Whenever mistakes are made in the application of these rules, it is best to correct them by such discussion as in the paragraph above, rather than by appeal to the memorized rule.
>
> Multiplication of binomials is needed as preparation for the solution of quadratic equations. Using the method of the preceding paragraphs we consider $(x + 5)(x + 3)$. This gives us $x(x + 3) + 5(x + 3)$. The multiplication, $(2x - 7)(3x - 8)$, developed in this way leads directly to the rule for the sign of the product of two negative terms.

To some teachers this slow development may seem a waste of time. To me it seems the only way in which we can hope to trans-

form school algebra into a subject as thought-compelling, as educative, as we know how to make it.

The same sort of approach to the solution of equations will avoid the use of the word "transpose," which serves to short circuit the thought processes of the student, and replace this word by "additions" and "subtractions." If in an equation we have 10 too much in one member, the logical remedy is to subtract 10 from each member; if we are short 10, the logical remedy is to add 10 to each member.

A certain psychologist has said that of course we do not want the pupil laboriously to think of the additions and subtractions made in simplifying equations. But thinking of additions and subtractions is not noticeably more laborious than thinking of numbers that fly up plus and come down minus. Furthermore if by "laborious" the psychologist means thought-compelling, then that is exactly what we do want, and what we must have if we are to make algebra educative.

This brings us once more to the question of drill in the algebra class, its nature and its purposes. Very probably the success or failure, in this country, of the present movement toward a more intellectually stimulating presentation of algebra, depends upon what we come to think the high school student needs to do by way of drill.

What is drill? One way to prepare a pupil for a debate is to write out an argument and require him to go over it again and again. The hope is that repetitions enough will wear his brain paths so smooth that, even under the stress of the public debate, he will be able to speak his piece.

Another way to prepare a pupil for a debate is to discuss with him the arguments pro and con and so to guide his reading, thinking, and practice that he will seek out and think out his own defenses for and answers to each argument. The hope is that he will so thoroughly understand the subject and the arguments that he will be at home in dealing with any of them which may arise.

Each method has its advantages. The first surely produces immediate results. It enables the pupil to speak well on a subject even without understanding it very well. The second method is less showy at the outset but it has a richer intellectual flavor. In the case of my own son or of a student I was preparing for life in a democracy, I should prefer to have it used.

Doubtless most teachers admit at once the validity of the point I am trying to make. There are two kinds of drill. We readily slip away from the more important kind and content ourselves with the less important. The problem is a practical everyday problem, and not easy to solve. Whitehead says that the "necessary technical skill can only be acquired by a training which is apt to damage those energies of mind which should direct the technical skill." What we want to do is to avoid the damage. If the study of algebra is to become a more valuable mental experience, drill must be interpreted to include not only repetition of mechanical process but also repetition of explanation and of understanding—that repetition which comes from actual use in various situations at intervals of time. Drill should also include a repeated view of some little portion of the philosophy which underlies the subject. Let the fearful observe that understanding as well as mere repetition leads to skill; a skill which, though it may possibly develop more slowly at first, is likely to be less transitory. It may not even develop more slowly, because one may learn by insight in a few moments what it takes days to learn by repetition. In any event, the acquisition of skill at algebraic manipulation is less important than the acquisition of the inquiring and understanding habit of mind.

We must be on our guard against the attempt to project into algebra teaching the technique of drills which has been used by teachers of arithmetic. This attempt is based upon a misconception of the aims of secondary education and of the place in it which mathematics can play. It is belittling to our profession to conceive of the education of the adolescent as shutting him up in school for hours each day and giving him merely drills for skill.

Perhaps the greatest service which can be rendered to the teaching of algebra is to change the connotation of the word *drill*. It should include exercises in investigation, exploration, and discovery for the purpose of developing the scientific frame of mind. It should include exercises in explaining and understanding for the purpose of developing the habit of comprehending; drill in underlying principles as well as in superficial routine.

Contrast these two plans for teaching: (1) a routine of showing how, followed by mechanical drilling; and (2) the use of every topic so as to get from it the maximum of intellectual stimulus. The first is mere training in the narrowest sense; the second is

education. The first is teaching algebra; the second is teaching children. The second enables us to teach more *about* algebra, and it shows that we need less drill of the deadening sort than we used to think was needed.

To free the teacher. Progress comes from the obstinate individual who refuses to take the easiest way. It does not come from the obstinate individual who refuses to take any but his old way. Nor is it sure to come from the flighty individual who seeks always some new way. It comes from the individual who "proves all things and holds fast to that which is good." This individual may be writer, supervisor, principal, head of department, or teacher in the ranks. The point is that progress comes from the individual. Hence the individual must be free.

In a great city we have waged long warfare to free the junior high school grades from those standardized tests which give too meager a definition of the outcomes of teaching algebra. No uniform test should be given which is not designed to improve the teaching of algebra as well as to test it. The secondary effects of uniform examinations are more important than their primary, or testing, effects. When questions set by the College Entrance Examination Board in 1912 are still being labored over in 1932; when thousands of schools will teach what is set by the Board and will not teach what is not set by the Board, we can clearly see the tremendous power which this secondary effect of uniform examinations gives to examining bodies, and the tremendous responsibility. The College Board should not, I think, set any question which is not worth while in itself and which does not direct teaching aright. Its algebra papers taken as a whole should give evidence that algebra connects with important fields of thought, that it has methods of doing important things, that its applications are significant in modern life.

The teacher should be freed so far as possible from hampering tests: tests which turn his energies to the development of skill of so many kinds as to leave no time for the development of higher powers. This does not mean that there should be no supervision. Old, experienced, and wise teachers can best serve in the guidance of young and inexperienced teachers. Their supervision should be sympathetic, not so much seeking conformity as letting the young teacher, in the first place, profit by the lore of his older colleague and then letting him build his own methods and devices.

I do not mean that there should be no uniform tests. Such tests may be necessary in some cases to give tone to the work. Not only so, but such tests may be used with great effect by the skilled administrator in directing teaching into channels that are right.

But when all the demands of supervision have been met; when all uniform tests have been prepared for; and when all courses of study have been complied with, there should still remain time in which the teacher can try his own ideas, make his own experiments; time in which he no longer need hasten to impart mere knowledge but can make knowledge become in the minds of his pupils a living thing.

If we are to be free we must be worthy to be free. I would speak first of our education. The National Committee suggests, as minimum preparation, courses in elementary analytics and in the calculus. To these, physics should surely be added, because in its field are found so many of the simple and interesting applications of secondary school mathematics.

While we cannot yet hope to be as advantageously situated as are the mathematics teachers of the public schools of England, who themselves teach courses in mechanics, thus getting theory and application side by side; while we have not yet begun to put into effect the suggestion made by Professor Moore in his famed presidential address of 1901, that we unify our physics and mathematics (see the reprint of his address in the *First Yearbook* of the Council); yet we can familiarize ourselves with the field of elementary physics and put ourselves in a position to profit by the present great interest in science, and bring into our discussions of the classroom one more touch of reality. Algebra, geometry, and trigonometry will all be gainers thereby. It is greatly to be desired that as our educational preparation continues to improve we shall include in it not a few courses in science.

Then again we should test our teaching by that best of all tests—the attitude of our pupils. Are they interested? Do they expect to understand the "why" and the "what for" or only the "how"? Do they have the habit of expecting to understand? Are they learning to organize their ideas and express them cogently? Are they getting width of view and range of application? Are they learning to connect a certain sphere of human activity with the scientific principles upon which its successful conduct depends?

If we teachers are to be worthy to be free we must have some

vision of the greatness of our opportunity. We know that in our daily contacts with the young people we are shaping their minds, molding their mental habits; that as we lead them in our mathematics classes into a scientific attitude of mind, as we lead them from special ideas to general ideas, we are really educating them, and we are in a very real sense creating the nation of to-morrow.

Conclusion. For many obvious reasons the present time offers unusual opportunities in junior high school algebra. Foremost among them is the opportunity to get the student increasingly to think.

We think clearly and definitely only when we have a problem which compels such thinking. Ordinarily our mental processes go on without much effort and largely in a mechanical manner, but this process is changed as soon as some obstacle is placed in our path, and the greater this obstacle is, provided that it does not plunge us into a state of emotional confusion, but is capable of being surmounted, the more developed the thought and the richer the content of our mental states. If we want to make the student think, we confront him with some difficulty, something that breaks up for the time his habitual modes of association and behavior.

The kind of thinking that algebra is particularly able to foster is scientific thinking. It is analytical; it is fact finding; it is free from prejudice; it seeks necessary conclusions, refusing to be turned aside from the truth. It is the kind of thinking by which our Newtons and our Edisons have created the present age. It is the kind of thinking which we must hope to make in some measure a universal attribute if the problems of democracy are to be solved by education.

To mold the thinking of a people seems a great undertaking, yet it is accomplished in one generation by molding the thinking of the children. Our success or failure at it depends upon the daily devices of the classroom; upon our patience in approach; upon our concept of drill; upon our definition of algebra and what we aim at in its teaching.

And all these things depend upon you and me; upon the individual teacher.

METHODS OF TEACHING VERBAL PROBLEMS

By JOSEPH A. NYBERG

Hyde Park High School, Chicago, Illinois

This chapter discusses, first, a general method of solution applicable to all verbal problems; second, the order in which the various types of problems can be treated; and third, a method that uses mostly oral work.

I. A General Method

Many attempts have been made to improve the teaching of verbal problems and to find a method that can be used for all types. Even after a pupil has developed a certain amount of skill in solving *coin* problems he must be taught independently to solve *age* problems, and the teaching must be repeated when he reaches *work* problems. The general method described in this chapter has doubtless been used by many teachers, but it has not found a place in the literature of the subject. I shall present the method as it could be developed in the classroom.

Let us suppose that a pupil comes to class and says that he is unable to solve the following problem:

> Mary has 3 more dimes than nickels and has a total of \$1.65. How many nickels and how many dimes has she?

The teacher, instead of asking the customary questions leading to x nickels and $x + 3$ dimes and the value of these coins, asks the pupil to *guess what the answer might be.* And the pupil guesses, let us suppose, that *Mary has 6 nickels.* This is, of course, an incomplete answer because the number of dimes should also be stated. But the teacher can overlook this deficiency and ask, "Can you in some way test the answer to see if your guess is correct?" The pupil will very likely do some mental arithmetic or some multiplying at the board and announce that his answer is wrong. The teacher's task now consists in making the pupil tell exactly what he did to find out if his guess is right or wrong. While this may

seem easy to do, we know that pupils without proper training in talking before a class frequently cannot state just what they have done. Perhaps the pupil will say, "Well, 5 times 6 is 30, and 10 times 9 is 90, and that makes $1.20, which is wrong." The teacher must insist on having the pupil write on the blackboard exactly what he did.

T. Where did you get the number 5?

P. A nickel is 5 cents, and 6 nickels are 30 cents.

T. Instead of writing the number 30 to show the value of the nickels, write 5×6 so that everyone in the class can see exactly what numbers you are using.

After the pupil has done this, the teacher continues in the following manner:

T. You said something about a number 9. How did you get the number 9?

P. If there are 6 nickels there are 9 dimes because Mary has 3 more dimes than nickels.

T. But *how* did you get the number 9? Did you add, subtract, multiply, or . . . ?

P. I added 3 to 6.

T. Instead of writing 9, write $6 + 3$ so that no one will ask later where you got the 9. Now, what did you do with the number 9?

P. I got 90 cents because 9 dimes are worth 90 cents.

T. But *how* did you get the 90? Did you add, subtract, multiply, divide, or . . . ?

P. I took 10 times 9.

T. Then write 10 times the quantity 6 plus 3. Instead of doing the additions and multiplications, we will merely *indicate* them. After that, what did you do?

P. I added 30 and 90.

T. You mean you added 5 times 6 to 10 times the quantity 6 plus 3. Write it in that way so that the class can see exactly what happens to every number. If your guess had been correct, this quantity would equal what number?

In this manner the teacher must get the pupil to write the following statement:

$$5 \times 6 + 10(6 + 3) = 165.$$

As a matter of good training, the teacher should also insist that all

the pupil's analysis be written properly as shown in the following:

The number of nickels	$= 6$
The number of dimes	$= 6 + 3$
The value of the nickels	$= 5 \times 6$
The value of the dimes	$= 10(6 + 3)$

We are now ready for the significant step in this method.

T. Now pick up an eraser; erase your guess, 6, wherever it appears in your work; and substitute the number x for the number 6.

After this substitution the pupil sees that his work presents the same appearance as it would have if he had worked the problem in the traditional manner, and he has found the desired equation

$$5x + 10(x + 3) = 165.$$

Evidently the method consists in (1) guessing an answer, (2) indicating the arithmetic work necessary to check the guess, and (3) substituting x for the guess. But the arithmetic work must *not* be performed; it must only be indicated. Otherwise the pupil will be unable to substitute the number x for the number guessed. Hence when developing this method in class, it is well to precede the work by some remarks about *indicating arithmetic operations* instead of performing them.

The success of this method depends not only on indicating an operation instead of performing it but also on the pupil's ability to solve certain problems that are entirely arithmetic in character. In the preceding illustration, for example, the pupil would fail if he could not solve the following problem:

What is the value in cents of 6 nickels and 9 dimes?

We can therefore expect to have trouble with those algebraic problems in which the corresponding arithmetic problem has not been used in previous grades. The *coin* and the *time-rate-distance* problems will give little difficulty because the pupil is familiar with money and with traveling, in a motor car if in no other way. The *age*, *mixture*, and *work* problems are the ones in which the arithmetic situation needs discussion.

Age problems, whether treated by the method here suggested or by any other method, lead to errors for the simple reason that the pupil overlooks the fact that when A gets 15 years older, for ex-

ample, the same fate is in store for B. After this fact has been sufficiently emphasized, the chief difficulty begins to disappear. To the adult it is perfectly obvious that he is getting older each year, but the pupil needs to be reminded of it again and again and again. Of course we could remedy this by changing the wording of a problem from:

> A is 15 years older than B. Three years from now A will be twice as old as B. Find their ages now.

and make it read:

> A is 15 years older than B. Three years from now A will be twice as old as B will be 3 years from now. Find their ages now.

Or, to make it still easier:

> A is 15 years older than B. A's age 3 years from now will be two times B's age 3 years from now.

With *mixture* problems of the type—

> How many pounds of 50-cent coffee must be mixed with 60 pounds of 40-cent coffee to get a mixture worth 48 cents a pound?

the pupil needs to be told that the grocer expects the receipts from the sale of the mixture to equal his receipts from the sales of the two varieties. We should not expect the pupil to know this fact. The pupil would be correct in thinking that the grocer is rather a stupid creature if he does not make an extra profit from his work in mixing the coffees. This trouble can be remedied by changing the problem to read:

> A country grocer has two kinds of coffee, one kind worth 50 cents a pound and the other worth 40 cents a pound. He has just 60 pounds of the cheaper kind which he cannot sell because the millworkers in his neighborhood have received a raise in wages. So he decides to mix the 60 pounds of 40 cent coffee with some of the 50 cent coffee and retail it all at 48 cents a pound. He will be so pleased to get rid of the cheaper coffee that he will not try to make an extra profit by the deal, etc.

Or, the problem could state that for a special bargain day he plans to mix the coffees, and expects to get as much money from the sales of the mixture as he would get by selling each kind sep-

arately. Note again that the pupil's trouble is not with the algebra but in understanding the situation described in the problem. And these situations can be very well explained by working a few problems of arithmetic, such as finding the average value when 30 pounds at 40 cents a pound are mixed with 35 pounds at 45 cents a pound.

Consider next a *work* problem such as:

> John can mow a lawn in 50 minutes and Henry can do it in 40 minutes. After John has been working 15 minutes Henry begins to help him. How many minutes are needed to finish the work after Henry begins?

When the pupil first sees a problem of this type he is inclined to think that if one man can do a piece of work in 50 minutes and another man can do it in 40, then the average of the numbers, or 45, is the number of minutes needed when the two work together. Since this can easily be shown to be wrong, the pupil feels that half the average, or 22½, would be a reasonable time for the two men.

Consider the simpler problem:

> John can saw some wood in 4 hours and Henry can do the same work in 3 hours. How many hours will it take them when working together?

No pupil can work this problem until he sees that the important item is the fractional part of the work done in one unit of time. I doubt whether this idea would ever occur to any pupil without a suggestion from the teacher. Here again the pupil's difficulty is not with algebra but with arithmetic.

Since by this method of teaching verbal problems the difficulties are transferred from the field of algebra to that of arithmetic, we should undoubtedly have in the seventh and eighth grades more arithmetic problems of the following kind:

1. A boy has 3 nickels and 8 dimes. How much money has he?
2. A boy is 15 years old, and his sister is 12 years old. What is the ratio of their ages? What will be the ratio of their ages 9 years from now? What was the ratio of their ages 6 years ago?
3. A grocer mixes 10 pounds of coffee worth 50 cents a pound with 15 pounds of coffee worth 45 cents a pound. What is all the coffee worth? What is the average value per pound of the coffee?
4. A garage attendant has a solution which is 10% alcohol. To 5 gallons of this mixture he adds 4 gallons of water. What is the per cent of alcohol in his final mixture?

5. John can do some work in 5 hours and Henry can do the same work in 4 hours. How many hours would it take them to do the same work together?

6. Mr. Smith travels 4 hours at an average rate of 32 miles an hour, and then travels 3 hours at an average rate of 40 miles an hour. What is the total distance he has traveled? What is his average rate for the entire trip?

Since the intuitive geometry and other kinds of work are replacing these problems in the seventh and eighth grades, the algebra teacher should consider at least one problem of each of the various kinds before introducing the corresponding algebraic problem.

I have sketched this method of teaching verbal problems in order to show that a general method applicable to all types does exist, and with the hope that other teachers will also experiment with the method, and report on its good and weak features. I do not believe the method is any better than our traditional method, but I find that after a pupil has worked a few problems by the general method he can better understand what he is doing when he writes, for example,

Let the number of nickels $= x$.

The general method is like the ancient "Rule of False Position," a description of which is found in *The History and Significance of Certain Standard Problems in Algebra* by Vera Sanford, and which is illustrated by the problem:

> The head of a fish weighs ⅓ of the whole fish, his tail weighs ¼ of the whole fish, and his body weighs 30 ounces. What does the whole fish weigh?

The problem would be solved as follows: Suppose the whole fish weighs 12 ounces. Then the head would weigh 4, the tail 3, and the body 5 ounces. Evidently the required weight of the whole fish is the same multiple of 12 that 30 is of 5. Hence the fish weighs 6 times 12, or 72 ounces.

Concerning this method Sanford says (page 96): "The student who uses some arithmetic device comparable to the Rule of False Position probably does thinking of higher grade than the one who mechanically lets n stand for the unknown quantity, sets up the equation, and solves it."

II. The Order of the Problems

If one problem leads to a quadratic equation for its solution, and another problem leads to a linear equation, there is no question about the order in which these problems should be treated. Hence we shall deal only with problems leading to linear equations. Again, there is no question of the order if one problem involves parentheses and another problem does not. Except for these obvious cases, I can find, in looking over the literature of the subject, only one article ("Systematic Procedure in the Solution of Algebraic Problems," by R. R. Goff, *The Mathematics Teacher*, October, 1923, pp. 350-55) which discusses the best order in which to treat the problems. The problems are there divided into three cases.

Case 1. The problem has two unknowns, the second of which can be derived from the first.

> The sum of two numbers is 63 and their difference is 17. Find the numbers.

Case 2. The problem has two unknowns from which two other unknowns are derived, and the equation is made from the derived unknowns.

> *A* is three times as old as *B*, but in ten years he will be only twice as old. Find their ages.

Case 3. The derived unknowns are not obtained from any statement of the problem but are implied, and the equation is made from the derived unknowns.

> $1,000 is divided in two parts, the first part invested at 6% and the second part at 5%. If the total annual income is $580, find each part.

About 1920 I began experimenting with an order of problems which proved so successful that I have never felt any need of changing it. The order is used in my text published in 1924—but the ideas on which it is based are not evident from a casual inspection of the book. The problems are divided into the following types and studied in this order:

Type 1. There is only one unknown and the equation is found by a direct translation of the English words into an algebraic equation.

> Five added to six times a certain number equals 29. Find the number.

Type 2. The unknowns are x and ax. The equation is $x + ax = b$. (a and b are integers, positive or negative.)

John has 6 times as many books as Henry. Together they have 35 books. How many books has each?

Type 3. The unknowns are x and $x + a$. The equation is $x + x + a = b$.

One tank holds 3 gallons more than another tank. Their combined capacity is 25 gallons. Find the capacity of each tank.

At first glance it might seem that the order of Types 2 and 3 is immaterial, but Type 2 is simpler. In the third type there are two ways of beginning the problem: the pupil can use x and $x + a$ for the unknowns or x and $x - a$. There is no objection to this; in fact, both methods should be discussed in class. Since only one method can be used in Type 2, it should precede Type 3.

Type 4. The unknowns are x and ax, or x and $x + a$. The equation is $bx + c(x + a) = d$.

A boy has 2 more dimes than he has nickels. If he has 80 cents, how many dimes and how many nickels has he?

This type differs from the others in that the original unknowns are multiplied by some number before the equation can be formed. It is the same as Case 2 in Goff's classification. It is not introduced until the pupil can solve equations containing parentheses.

Type 5. The unknowns are x and $ax + b$. The equation can be any one of the previous types: $x + ax + b = c$ or $cx + d(ax + b) = e$.

In this type of problem the word *exceeds* and its synonyms are used.

One number exceeds twice another number by 5. Find the numbers if 13 times the smaller is 30 more than 4 times the larger.

Type 6. The unknowns are x and $s - x$. The equation is $ax + b(s - x) = c$.

A grocer has some 60-cent coffee and some 40-cent coffee. How many pounds of each kind should be used to make 15 pounds of a mixture worth 52 cents a pound?

The above classification is based on the nature of the algebraic work in the problem regardless of whether the problem deals with